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7-5 Study Guide and Intervention

Roots and Zeros

Types of Roots The following statements are equivalent for any polynomial function $f(x)$.

- c is a zero of the polynomial function $f(x)$.
- $(x - c)$ is a factor of the polynomial $f(x)$.
- c is a root or solution of the polynomial equation $f(x) = 0$.

If c is real, then $(c, 0)$ is an intercept of the graph of $f(x)$.

Fundamental Theorem of Algebra	Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.
Corollary to the Fundamental Theorem of Algebras	A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.
Descartes' Rule of Signs	If $P(x)$ is a polynomial with real coefficients whose terms are arranged in descending powers of the variable, • the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and • the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

Example 1 **Solve the equation $6x^3 + 3x = 0$ and state the number and type of roots.**

$6x^3 + 3x = 0$

$3x(2x^2 + 1) = 0$

Use the Zero Product Property.

$3x = 0$ or $2x^2 + 1 = 0$

$x = 0$ or $2x^2 = -1$

$x = \pm\frac{i\sqrt{2}}{2}$

The equation has one real root, 0, and two imaginary roots, $\pm\frac{i\sqrt{2}}{2}$.

Example 2 **State the number of positive real zeros, negative real zeros, and imaginary zeros for $p(x) = 4x^4 - 3x^3 + x^2 + 2x - 5$.**

Since $p(x)$ has degree 4, it has 4 zeros.

Use Descartes' Rule of Signs to determine the number and type of real zeros. Since there are three sign changes, there are 3 or 1 positive real zeros.

Find $p(-x)$ and count the number of changes in sign for its coefficients.

$p(-x) = 4(-x)^4 - 3(-x)^3 + (-x)^2 + 2(-x) - 5$

$= 4x^4 + 3x^3 + x^2 - 2x - 5$

Since there is one sign change, there is exactly 1 negative real zero.

Exercises

Solve each equation and state the number and type of roots.

1. $x^2 + 4x - 21 = 0$
2. $2x^3 - 50x = 0$
3. $12x^3 + 100x = 0$

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

4. $f(x) = 3x^3 + x^2 - 8x - 12$
5. $f(x) = 2x^4 - x^3 - 3x + 7$
6. $f(x) = 3x^5 - x^4 - x^3 + 6x^2 - 5$

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7-5 Study Guide and Intervention *(continued)*

Roots and Zeros

Find Zeros

Complex Conjugate Theorem	Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example **Find all of the zeros of $f(x) = x^4 - 15x^2 + 38x - 60$.**

Since $f(x)$ has degree 4, the function has 4 zeros.

$f(x) = x^4 - 15x^2 + 38x - 60 \quad f(-x) = x^4 - 15x^2 - 38x - 60$

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there is 1 sign change for the coefficients of $f(-x)$, the function has 1 negative real zero. Use synthetic substitution to test some possible zeros.

2	1	0	−15	38	−60
		2	4	−22	32
	1	2	−11	16	−28
3	1	0	−15	38	−60
		3	9	−18	60
	1	3	−6	20	0

So 3 is a zero of the polynomial function. Now try synthetic substitution again to find a zero of the depressed polynomial.

−2	1	3	−6	20
		−2	−2	16
	1	1	−8	36
−4	1	3	−6	20
		−4	4	8
	1	−1	−2	28
−5	1	3	−6	20
		−5	10	−20
	1	−2	4	0

So -5 is another zero. Use the Quadratic Formula on the depressed polynomial equation $x^2 - 2x + 4 = 0$ to find the other 2 zeros, $1 \pm i\sqrt{3}$.

The function has two real zeros at 3 and -5 and two imaginary zeros at $1 \pm i\sqrt{3}$.

Exercises

Find all of the zeros of each function.

1. $f(x) = x^3 + x^2 + 9x + 9$

2. $f(x) = x^3 - 3x^2 + 4x - 12$

3. $p(a) = a^3 - 10a^2 + 34a - 40$

4. $p(x) = x^3 - 5x^2 + 11x - 15$

5. $f(x) = x^3 + 6x + 20$

6. $f(x) = x^4 - 3x^3 + 21x^2 - 75x - 100$

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7-6 Study Guide and Intervention

Rational Zero Theorem

Lesson 7-6

Identify Rational Zeros

Rational Zero Theorem	Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_n and q is a factor of a_0.
Corollary (Integral Zero Theorem)	If the coefficients of a polynomial are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.

Example **List all of the possible rational zeros of each function.**

a. $f(x) = 3x^4 - 2x^2 + 6x - 10$

If $\frac{p}{q}$ is a rational root, then p is a factor of -10 and q is a factor of 3. The possible values for p are ±1, ±2, ±5, and ±10. The possible values for q are ±1 and ±3. So all of the possible rational zeros are $\frac{p}{q} = \pm1, \pm2, \pm5, \pm10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}$, and $\pm\frac{10}{3}$.

b. $q(x) = x^3 - 10x^2 + 14x - 36$

Since the coefficient of x^3 is 1, the possible rational zeros must be the factors of the constant term -36. So the possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, and ±36.

Exercises

List all of the possible rational zeros of each function.

1. $f(x) = x^3 + 3x^2 - x + 8$

2. $g(x) = x^5 - 7x^4 + 3x^2 + x - 20$

3. $h(x) = x^4 - 7x^3 - 4x^2 + x - 49$

4. $p(x) = 2x^4 - 5x^3 + 8x^2 + 3x - 5$

5. $q(x) = 3x^4 - 5x^3 + 10x + 12$

6. $r(x) = 4x^5 - 2x + 18$

7. $f(x) = x^7 - 6x^5 - 3x^4 + x^3 + 4x^2 - 120$

8. $g(x) = 5x^6 - 3x^4 + 5x^3 + 2x^2 - 15$

9. $h(x) = 6x^5 - 3x^4 + 12x^3 + 18x^2 - 9x + 21$

10. $p(x) = 2x^7 - 3x^6 + 11x^5 - 20x^2 + 11$

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7-6 Study Guide and Intervention *(continued)*

Rational Zero Theorem

Find Rational Zeros

Example 1 **Find all of the rational zeros of $f(x) = 5x^3 + 12x^2 - 29x + 12$.**

From the corollary to the Fundamental Theorem of Algebra, we know that there are exactly 3 complex roots. According to Descartes' Rule of Signs there are 2 or 0 positive real roots and 1 negative real root. The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±12, $\pm\frac{1}{5}$, $\pm\frac{2}{5}$, $\pm\frac{3}{5}$, $\pm\frac{4}{5}$, $\pm\frac{6}{5}$, $\pm\frac{12}{5}$. Make a table and test some possible rational zeros.

$\frac{p}{q}$	5	12	−29	12
1	5	17	−12	0

Since $f(1) = 0$, you know that $x = 1$ is a zero.
The depressed polynomial is $5x^2 + 17x - 12$, which can be factored as $(5x - 3)(x + 4)$.
By the Zero Product Property, this expression equals 0 when $x = \frac{3}{5}$ or $x = -4$.
The rational zeros of this function are 1, $\frac{3}{5}$, and -4.

Example 2 **Find all of the zeros of $f(x) = 8x^4 + 2x^3 + 5x^2 + 2x - 3$.**

There are 4 complex roots, with 1 positive real root and 3 or 1 negative real roots. The possible rational zeros are ±1, ±3, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, $\pm\frac{1}{8}$, $\pm\frac{3}{2}$, $\pm\frac{3}{4}$, and $\pm\frac{3}{8}$.

Make a table and test some possible values.

$\frac{p}{q}$	8	2	5	2	−3
1	8	10	15	17	14
2	8	18	41	84	165
$\frac{1}{2}$	8	6	8	6	0

Since $f\left(\frac{1}{2}\right) = 0$, we know that $x = \frac{1}{2}$ is a root.

The depressed polynomial is $8x^3 + 6x^2 + 8x + 6$.
Try synthetic substitution again. Any remaining rational roots must be negative.

$\frac{p}{q}$	8	6	8	6
$-\frac{1}{4}$	8	4	7	$4\frac{1}{4}$
$-\frac{3}{4}$	8	0	8	0

$x = -\frac{3}{4}$ is another rational root.
The depressed polynomial is $8x^2 + 8 = 0$, which has roots $\pm i$.

The zeros of this function are $\frac{1}{2}$, $-\frac{3}{4}$, and $\pm i$.

Exercises

Find all of the rational zeros of each function.

1. $f(x) = x^3 + 4x^2 - 25x - 28$

2. $f(x) = x^3 + 6x^2 + 4x + 24$

Find all of the zeros of each function.

3. $f(x) = x^4 + 2x^3 - 11x^2 + 8x - 60$

4. $f(x) = 4x^4 + 5x^3 + 30x^2 + 45x - 54$