*n*th Roots and Rational Exponents

Explore Inverses of Power Functions

Online Activity Use a calculator to complete the Explore.

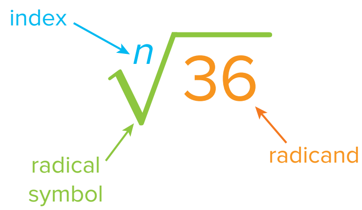
INQUIRY What conjectures can you make about $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ for all odd positive values of n?

Learn *n*th Roots

Finding the square root of a number and squaring a number are inverse operations. To find the square root of a, you must find a number with a square of a. The inverse of raising a number to the nth power is finding the ***n*th root** of a number. The symbol $\sqrt[n]{}$ indicates an nth root.

For any real numbers a and b and any positive integer n, if $a^n = b$, then a is an nth root of b. For example, because $(-2)^6 = 64$, -2 is a sixth root of 64 and 2 is a principal root.

An example of an nth root is $\sqrt[n]{36}$, which is read as *the nth root of 36*. In this example, n is the **index** and 36 is the **radicand**, or the expression under the radical symbol.



Some numbers have more than one real nth root. For example, 16 has two square roots, 4 and -4, because 4^2 and $(-4)^2$ both equal 16. When there is more than one real root and n is even, the nonnegative root is called the **principal root**.

Key Concept • Real *n*th Roots

Suppose n is an integer greater than 1, a is a real number, and a is an nth root of b.

a	***n* is even.**	***n* is odd.**
$a > 0$	1 unique positive and 1 unique negative real root: $\pm\sqrt[n]{a}$	1 unique positive and 0 negative real root: $\sqrt[n]{a}$
$a < 0$	0 real roots	0 positive and 1 negative real root: $\sqrt[n]{a}$
$a = 0$	1 real root: $\sqrt[n]{0} = 0$	1 real root: $\sqrt[n]{0} = 0$

A radical expression is simplified when the radicand contains no fractions and no radicals appear in the denominator.

Today's Goals

- Simplify expressions involving radicals and rational exponents.
- Simplify expressions in exponential or radical form.

Today's Vocabulary

*n*th root
index
radicand
principal root
rational exponent

Think About It!

Lorena says she can tell that $\sqrt[3]{-64}$ will have a real root without graphing. Do you agree or disagree? Explain your reasoning.

Agree; sample answer: Because n is odd and a is less than 0, I know that the expression will have one negative real root.

Your Notes

Watch Out!

Principal Roots
Because $5^2 = 25$, 25 has two square roots, 5 and -5. However, the value of $\sqrt{25}$ is 5 only. To indicate both square roots and not just the principal square root, the expression must be written as $\pm\sqrt{25}$.

Talk About It!

Compare the simplified expressions in the previous example with the ones in this example. Explain why the simplified expressions in this example require absolute value bars when the simplified expressions in the previous example did not.

Sample answer: None of the expressions in the previous example resulted in an odd power when taking an even root of an even power. However, both expressions in this example did result in an odd power when taking an even root of an even power.

Example 1 Find Roots

Simplify.

a. $\pm\sqrt{25x^4}$

$$\pm\sqrt{25x^4} = \pm\sqrt{(\underline{5x^2})^2}$$
$$= \pm\ \underline{5x^2}$$

b. $-\sqrt{(y^2+7)^{12}}$

$$-\sqrt{(y^2+7)^{12}} = -\sqrt{[(y^2+7)^6]^2}$$
$$= -(y^2+7)^6$$

c. $\sqrt[3]{343a^{18}b^6}$

$$\sqrt[3]{343a^{18}b^6} = \sqrt[3]{(\underline{7a^6b^2})^3}$$
$$= \underline{7a^6b^2}$$

d. $\sqrt{-289c^8d^4}$

There are no real roots of $\sqrt{-289}$. However, there are two imaginary roots, $17i$ and $-17i$. Because we are only finding the principal square root, use $17i$.

$$\sqrt{-289c^8d^4} = \sqrt{\underline{-1}} \cdot \sqrt{289c^8d^4}$$
$$= \underline{i} \cdot \sqrt{289c^8d^4}$$
$$= \underline{17ic^4d^2}$$

Check

Write the simplified form of each expression.

a. $\pm\sqrt{196x^4}$ $\underline{\pm 14x^2}$ **b.** $-\sqrt{196x^4}$ $\underline{-14x^2}$ **c.** $\sqrt{-196x^4}$ $\underline{14ix^2}$

When you find an even root of an even power and the result is an odd power, you must use the absolute value of the result to ensure that the answer is nonnegative.

Example 2 Simplify Using Absolute Value

Simplify.

a. $\sqrt[4]{81x^4}$

$$\sqrt[4]{81x^4} = \sqrt[4]{(3x)^4}$$
$$= 3|x|$$

Because x could be negative, you must use the absolute value of x to ensure that the principal square root is nonnegative.

b. $\sqrt[8]{256(y^2-2)^{24}}$

$$\sqrt[8]{256(y^2-2)^{24}} = \sqrt[8]{\underline{256}} \cdot \sqrt[8]{(y^2-2)^{24}}$$
$$= \underline{2}\,|(y^2-2)^3|$$

Because $(y^2-2)^3$ could be negative, you must use the absolute value of $(y^2-2)^3$ to ensure that the principal square root is nonnegative.

Learn Rational Exponents

You can use the properties of exponents to translate expressions from exponential form to radical form or from radical form to exponential form. An expression is in **exponential form** if it is in the form x^n, where n is an exponent. An expression is in **radical form** if it contains a radical symbol.

For any real number b and a positive integer n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, except where $b < 0$ and n is even. When $b < 0$ and n is even, a complex root may exist.

Examples: $125^{\frac{1}{3}} = \sqrt[3]{125}$ or 5 $\qquad (-49)^{\frac{1}{2}} = \sqrt{-49}$ or $7i$

The expression $b^{\frac{1}{n}}$ has a **rational exponent**. The rules for exponents also apply to rational exponents.

Key Concept • Rational Exponents

For any nonzero number b and any integers x and y, with $y > 1$, $b^{\frac{x}{y}} = \sqrt[y]{b^x} = \left(\sqrt[y]{b}\right)^x$, except when $b < 0$ and y is even. When $b < 0$ and y is even, a complex root may exist.

Examples: $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2$ or 25

$(-49)^{\frac{3}{2}} = \left(\sqrt{-49}\right)^3 = (7i)^3$ or $-343i$

Key Concept • Simplest Form of Expressions with Rational Exponents

An expression with rational exponents is in simplest form when all of the following conditions are met.

- It has no negative exponents.
- It has no exponents that are not positive integers in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

Example 3 Radical and Exponential Forms

Simplify.

a. Write $x^{\frac{4}{3}}$ in radical form.

$x^{\frac{4}{3}} = \sqrt[3]{x^4}$

b. Write $\sqrt[5]{x^2}$ in exponential form.

$\sqrt[5]{x^2} =$ $x^{\frac{2}{5}}$

Go Online You can complete an Extra Example online.

Math History Minute:

Christoff Rudolff (1499–1543) wrote the first German algebra textbook. It is believed that he introduced the radical symbol $\sqrt{}$ in 1525 in his book *Die Coss*. Some feel that this symbol was used because it resembled a small *r*, the first letter in the Latin word *radix* or root.

Think About It!

Write two equivalent expressions, one in radical form and one in exponential form.

Sample answer:

$\sqrt[4]{16}$, $16^{\frac{1}{4}}$

Think About It!

Why did you set t equal to $\frac{1}{4}$?

Sample answer: t is the time in years, and 3 months is equal to $\frac{1}{4}$ of a year.

Think About It!

How can you tell if $x^{\frac{y}{z}}$ will simplify to an integer?

Sample answer: If you can rewrite x as an integer to the power of z, it will simplify to an integer

Go Online

to see more examples of evaluating expressions with rational exponents.

Watch Out!

Exponents Recall that when you multiply powers, the exponents are added, and when you raise a power to a power, the exponents are multiplied.

Think About It!

How would the expression in part **c** change if the exponents were $\frac{1}{3}$ and $-\frac{3}{4}$?

Sample answer: The expression would not have a positive exponent, so I would need to follow the process used in part **b**.

Example 4 Use Rational Exponents

FINANCIAL LITERACY The expression $c(1+r)^t$ can be used to estimate the future cost of an item due to inflation, where c represents the current cost of the item, r represents the annual rate of inflation, and t represents the time in years. Write the expression in radical form for the future cost of an item 3 months from now if the annual rate of inflation is 4.7%.

$c(1+r)^t = c(1+0.047)^{\frac{1}{4}}$ — $r = 0.047$, $t = \frac{1}{4}$

$= c(\underline{1.047})^{\frac{1}{4}}$ — Add.

$= c\sqrt[4]{1.047}$ — $b^{\frac{1}{n}} = \sqrt[n]{b}$

Example 5 Evaluate Expressions with Rational Exponents

Evaluate $32^{-\frac{2}{5}}$.

$32^{-\frac{2}{5}} = \dfrac{1}{32^{\frac{2}{5}}}$ — $b^{-n} = \frac{1}{b^n}$

$= \dfrac{1}{(2^5)^{\frac{2}{5}}}$ — $32 = 2^5$

$= \dfrac{1}{2^{5\cdot\frac{2}{5}}}$ — Power of a Power

$= \underline{\dfrac{1}{2^2}}$ or $\dfrac{1}{4}$ — Multiply the exponents.

Example 6 Simplify Expressions with Rational Exponents

Simplify each expression.

a. $x^{\frac{2}{3}} \cdot x^{\frac{1}{6}}$

$x^{\frac{2}{3}} \cdot x^{\frac{1}{6}} = x^{\frac{2}{3}+\frac{1}{6}}$ — Add powers.

$= x^{\frac{4}{6}+\frac{1}{6}}$ — $\frac{2}{3} = \frac{4}{6}$

$= x^{\frac{5}{6}}$ — Add the exponents.

b. $y^{-\frac{2}{3}}$

$y^{-\frac{2}{3}} = \dfrac{1}{y^{\frac{2}{3}}}$ — $b^{-n} = \frac{1}{b^n}$

$= \dfrac{1}{y^{\frac{2}{3}}} \cdot \dfrac{y^{\frac{1}{3}}}{y^{\frac{1}{3}}}$ — $\frac{y^{\frac{1}{3}}}{y^{\frac{1}{3}}} = 1$

$= \dfrac{y^{\frac{1}{3}}}{y^{\frac{3}{3}}}$ or $\underline{\dfrac{y^{\frac{1}{3}}}{y}}$ — $y^{\frac{2}{3}} \cdot y^{\frac{1}{3}} = y^{\frac{2}{3}+\frac{1}{3}}$

c. $z^{-\frac{1}{3}} \cdot z^{\frac{3}{4}}$

$z^{-\frac{1}{3}} \cdot z^{\frac{3}{4}} = z^{-\frac{1}{3}+\frac{3}{4}}$ — Add powers.

$= \underline{z^{-\frac{4}{12}+\frac{9}{12}}}$ or $z^{\frac{5}{12}}$ — $-\frac{1}{3} = \frac{4}{12}, \frac{3}{4} = \frac{9}{12}$

Go Online You can complete an Extra Example online.