Operations on Functions

Today's Goals
- Find sums, differences, products, and quotients of functions.
- Find compositions of functions.

Today's Vocabulary
composition of functions

Explore Adding Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY Do you think that the graph of $f(x) + g(x)$ will be more or less steep than the graphs of $f(x)$ and $g(x)$?

Learn Operations on Functions

Key Concept • Operations on Functions

Operation	Definition	Example: Let $f(x) = 3x$ and $g(x) = 2x - 4$.
Addition	$(f + g)(x) = f(x) + g(x)$	$(f + g)(x) = 3x + (2x - 4)$ $= 5x - 4$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$(f - g)(x) = 3x - (2x - 4)$ $= x + 4$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(f \cdot g)(x) = 3x(2x - 4)$ $= 6x^2 - 12x$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$\left(\frac{f}{g}\right)(x) = \frac{3x}{2x-4}, x \neq 2$

To graph the sum or difference of functions, graph each function separately. Then add or subtract the corresponding functional values.

Example 1 Add and Subtract Functions

Given $f(x) = -x^2 + 3x + 1$ and $g(x) = 2x^2 - 5$, find each function.

a. $(f + g)(x)$

$(f + g)(x) = f(x) + \underline{g(x)}$ — Addition of functions

$= (-x^2 + 3x + 1) + (2x^2 - 5)$ — $f(x) = -x^2 + 3x + 1$ and $g(x) = 2x^2 - 5$

$= -x^2 + 3x + 1 + 2x^2 - 5$ — Add.

$= x^2 + \underline{3}x - \underline{4}$ — Simplify.

(*continued on the next page*)

Go Online You can complete an Extra Example online.

Study Tip

Degree If the degree of $f(x)$ is m and the degree of $g(x)$ is n, then the degrees of $(f + g)(x)$ and $(f - g)(x)$ can be at most m or n, whichever is greater.

Your Notes

b. $(f - g)(x)$

$(f - g)(x) = f(x) - \underline{g(x)}$ — Subtraction of functions

$= (\underline{-x^2 + 3x + 1}) - (2x^2 - 5)$ — $f(x) = -x^2 + 3x + 1$ and $g(x) = 2x^2 - 5$

$= -x^2 + 3x + 1 - 2x^2 + 5$ — Subtract.

$= -3x^2 + \underline{3}\,x + \underline{6}$ — Simplify.

Example 2 Multiply and Divide Functions

Given $f(x) = 4x^2 - 2x + 3$ and $g(x) = -x + 5$, find each function.

a. $(f \cdot g)(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$ — Multiplication of functions

$= (4x^2 - 2x + 3)(\underline{-x + 5})$ — $f(x) = 4x^2 - 2x + 3$ and $g(x) = -x + 5$

$= -4x^3 + \underline{20}\,x^2 + \underline{2}\,x^2 - 10x - \underline{3}\,x + 15$ — Distributive Property

$= -4x^3 + \underline{22}\,x^2 - \underline{13}\,x + 15$ — Simplify.

b. $\left(\frac{f}{g}\right)(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ — Division of functions

$= \frac{4x^2 - 2x + 3}{-x + 5}, x \neq \underline{5}$ — $f(x) = 4x^2 - 2x + 3$ and $g(x) = -x + 5$

Check

Given $f(x) = -x^2 + 1$ and $g(x) = x + 1$, find each function.

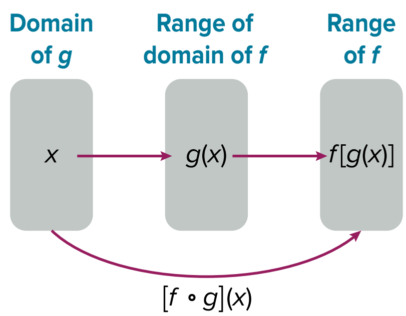
$(f \cdot g)(x) = \underline{-x^3 - x^2 + x + 1}$ $\quad\left(\frac{f}{g}\right)(x) = \underline{-x + 1}$

Go Online to see Example 3.

Learn Compositions of Functions

Key Concept • Composition of Functions

Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composition function $f \circ g$ can be described by $[f \circ g](x) = f[g(x)]$.



Go Online You can complete an Extra Example online.

Example 4 Compose Functions by Using Ordered Pairs

Given f and g, find $[f \circ g](x)$ and $[g \circ f](x)$. State the domain and range for each.

$f = \{(1, 12), (10, 11), (0, 13), (9, 7)\}$ $\quad$ $g = \{(4, 1), (5, 0), (13, 9), (12, 10)\}$

Part A Find $[f \circ g](x)$ and $[g \circ f](x)$.

To find $f \circ g$, evaluate $g(x)$ first then use the range to evaluate $f(x)$.

$f[g(4)] = f(1)$ or 12 $\quad$ $g(4) = 1$

$f[g(5)] = f(0)$ or 13 $\quad$ $g(5) = 0$

$f[g(13)] = f(9)$ or 7 $\quad$ $g(13) = 9$

$f[g(12)] = f(10)$ or 11 $\quad$ $g(12) = 10$

To find $g \circ f$, evaluate $f(x)$ first then use the range to evaluate $g(x)$.

$g[f(1)] = g(12)$ or 10 $\quad$ $f(1) = 12$

$g[f(10)] = g(11)$ $\quad$ $f(10) = 11$

$g[f(0)] = g(13)$ or 9 $\quad$ $f(0) = 13$

$g[f(9)] = g(7)$ $\quad$ $f(9) = 7$

Because 11 and 7 are not in the domain of g, $g \circ f$ is undefined for $x = 11$ and $x = 7$. So, $g \circ f = \{(1, 10), (0, 9)\}$.

Part B State the domain and range.

$[f \circ g](x)$: The domain is the x-coordinates of the composed function, so D = {4, 5, 12, 13}. The range is the y-coordinates of the composed function, so R = {7, 11, 12, 13}.

$[g \circ f](x)$: The domain is the x-coordinates of the composed function, so D = {0, 1}. The range is the y-coordinates of the composed function, so R = {9, 10}.

Example 5 Compose Functions

Given $f(x) = 2x - 5$ and $g(x) = 3x$, find $[f \circ g](x)$ and $[g \circ f](x)$. State the domain and range for each.

Part A Find $[f \circ g](x)$ and $[g \circ f](x)$.

$[f \circ g](x) = f[g(x)]$	Composition of functions	$[g \circ f](x) = g[f(x)]$
$= f(3x)$	Substitute.	$= g(2x - 5)$
$= 2(3x) - 5$	Substitute again.	$= 3(2x - 5)$
$= 6x - 5$	Simplify.	$= 6x - 15$

Part B State the domain and range.

Because $[f \circ g](x)$ and $[g \circ f](x)$ are both linear functions with nonzero slopes, D = {all real numbers} and R = {all real numbers} for both functions.

Go Online You can complete an Extra Example online.

Study Tip

Domain and Range To ensure you have the right domain and range, it can help to graph $[f \circ g](x)$ and $[g \circ f](x)$.

Check

Given $f(x) = -x + 1$ and $g(x) = 2x^3 - x$, find $[f \circ g](x)$ and $[g \circ f](x)$. State the domain and range for each.

$[f \circ g](x) = -2x^3 + x + 1$

Domain of $[f \circ g](x)$: all real numbers

Range of $[f \circ g](x)$: all real numbers

$[g \circ f](x) = -2x^3 + 6x^2 - 5x + 1$

Domain of $[g \circ f](x)$: all real numbers

Range of $[g \circ f](x)$: all real numbers

Apply Example 6 Use Composition of Functions

BOX OFFICE **A movie theater charges $8.50 for each of the *x* tickets sold. The manager wants to determine how much the movie theater gets to keep of the ticket sales if they have to give the studios 75% of the money earned on ticket sales *t*(*x*). If the amount they keep of each ticket sale is *k*(*x*), which composition represents the total amount of money the theater gets to keep?**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to write functions for $t(x)$ and $k(x)$ and use them to create a composition that represents the money that the theater keeps. If the studios get 75%, what does the theater get to keep? Should the composition be $[k \circ t](x)$ or $[t \circ k](x)$?

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: First, I will determine functions for $t(x)$ and $k(x)$. Then, I will determine the order of the composition and simplify it.

I will apply what I have learned in previous examples to complete the task.

3 What is your solution?

Use your strategy to solve the problem.

What function represents the money earned on ticket sales, $t(x)$? $t(x) = 8.50x$

What function represents the amount of money the theater keeps from each ticket sale, $k(x)$? $k(x) = 0.25x$

Because the theater uses the total earnings to determine the amount they keep from the ticket sales, what composition should be used to represent the situation? $[k \circ t] = 2.125x$

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: If the theater sells 1000 tickets in a weekend, they will earn $t(1000)$, or $8500. The theater will keep 25% of $8500, which is $k(8500) = \$2125$. This is the same value as $[k \circ t](1000)$.

Watch Out!

Order Remember that, for two functions $f(x)$ and $g(x)$, $[f \circ g](x)$ is not always equal to $[g \circ f](x)$. Given that the studios take their cut after the tickets have been sold, consider how that affects the order of $t(x)$ and $k(x)$.

Go Online

You can complete an Extra Example online.