Roots and Zeros

Today's Goals

- Use the Fundamental Theorem of Algebra to determine the numbers and types of roots of polynomial equations.
- Determine the numbers and types of roots of polynomial equations, find zeros, and use zeros to graph polynomial functions.

Today's Vocabulary

multiplicity

Explore Roots of Quadratic Polynomials

Online Activity Use graphing technology to complete the Explore.

INQUIRY Is the Fundamental Theorem of Algebra true for quadratic polynomials?

Learn Fundamental Theorem of Algebra

The zero of a function $f(x)$ is any value c such that $f(c) = 0$.

Key Concept • Zeros, Factors, Roots, and Intercepts

Words: Let $P(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of $P(x)$.
- c is a root or solution of $P(x) = 0$.
- $x - c$ is a factor of $a_n x^n + \ldots + a_1 x + a_0$.
- If c is a real number, then $(c, 0)$ is an x-intercept of the graph of $P(x)$.

Example: Consider the polynomial function $P(x) = x^2 + 3x - 18$.

The zeros of $P(x) = x^2 + 3x - 18$ are -6 and 3.

The roots of $x^2 + 3x - 18 = 0$ are -6 and 3.

The factors of $x^2 + 3x - 18$ are $(x + 6)$ and $(x - 3)$.

The x-intercepts of $P(x) = x^2 + 3x - 18$ are $(-6, 0)$ and $(3, 0)$.

Key Concept • Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Key Concept • Corollary to the Fundamental Theorem of Algebra

Words: A polynomial equation of degree n has exactly n roots in the set of complex numbers, including repeated roots.

Examples:

$2x^3 - 5x + 2$	$-x^4 + 2x^3 - 2x$	$x^5 - 6x^3 + x^2 - 1$
3 roots	4 roots	5 roots

Repeated roots can also be called roots of multiplicity m where m is an integer greater than 1. **Multiplicity** is the number of times a number is a zero for a given polynomial. For example, $f(x) = x^3 = x \cdot x \cdot x$ has a zero at $x = 0$ with multiplicity 3, because x is a factor three times. However, the graph of the function still only intersects the x-axis once at the origin.

Think About It!

What is the multiplicity of the zero at $x = 1$ for $p(x) = (x - 1)^5$? Explain your reasoning.

Sample answer: The zero at $x = 1$ has a multiplicity of 5 because $(x - 1)^5$ has $(x - 1)$ as a factor five times.

Your Notes

Study Tip

Repeated Roots If you factor a polynomial and a factor is raised to a power greater than 1, then there is a repeated root. The power to which the factor is raised indicates the multiplicity of the root. To be sure that you do not miss a repeated root, it can help to write out each factor. For example, you would write $x^2(x^2+49)$ as $x \cdot x(x^2+49)$ as a reminder that x^2 indicates a root of multiplicity 2.

Key Concept • Descartes' Rule of Signs

Let $P(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients and $a_0 \neq 0$. Then the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.

Example 1 Determine the Number and Type of Roots

Solve $x^4 + 49x^2 = 0$. State the number and type of roots.

$x^4 + 49x^2 = 0$		Original equation
$x^2(x^2 + 49) = 0$		Factor.
$x^2 = 0$	or $x^2 + 49 = 0$	Zero Product Property
$x = 0$	$x^2 = -49$	Subtract 49 from each side.
	$x = \pm\sqrt{-49}$	Square Root Property
	$x = \pm 7i$	Simplify.

The polynomial has degree 4, so there are four roots in the set of complex numbers. Because x^2 is a factor, $x = 0$ is a root with multiplicity 2, also called a double root. The equation has one real repeated root, 0, and two imaginary roots, $7i$ and $-7i$.

Example 2 Find the Number of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = x^5 - 2x^4 - x^3 + 6x^2 - 5x + 10$.

Because $f(x)$ has degree 5, it has five zeros, either real or imaginary. Use Descartes' Rule of Signs to determine the possible number and types of *real* zeros.

Part A Find the possible number of positive real zeros.

Count the number of changes in sign for the coefficients of $f(x)$.

$$f(x) = x^5 - 2x^4 - x^3 + 6x^2 - 5x + 10$$

yes, + to −; no, − to −; yes, − to +; yes, + to −; yes, − to +

There are 4 sign changes, so there are 4, 2, or 0 positive real zeros.

Part B Find the possible number of negative real zeros.

Count the number of changes in sign for the coefficients of $f(-x)$.

$$f(-x) = (-x)^5 - 2(-x)^4 - (-x)^3 + 6(-x)^2 - 5(-x) + 10$$
$$= -x^5 - 2x^4 + x^3 + 6x^2 + 5x + 10$$

no, − to −; yes, − to +; no, + to +; no, + to +; no, + to +

There is 1 sign change, so there is 1 negative real zero.

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Part C Find the possible number of imaginary zeros.

Positive Real Zeros	Negative Real Zeros	Imaginary Zeros	Total Zeros
4	1	0	$4 + 1 + 0 = 5$
2	1	2	$2 + 1 + 2 = 5$
0	1	4	$0 + 1 + 4 = 5$

Check

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 3x^6 - x^5 + 2x^4 + x^3 - 3x^2 + 13x + 1$. Write the rows in ascending order of positive real zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros
0	0	6
0	2	4
2	0	4
2	2	2
4	0	2
4	2	0

Talk About It!

If a polynomial has degree n and no real zeros, then how many imaginary zeros does it have? Explain your reasoning.

n; Sample answer: A polynomial that has degree n will have n complex zeros. If there are no real zeros, then it must have $n - 0 = n$ zeros.

Learn Finding Zeros of Polynomial Functions

Key Concept • Complex Conjugates Theorem

Words: Let a and b be real numbers, and $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example: If $1 + 2i$ is a zero of $f(x) = x^3 - x^2 + 3x + 5$, then $1 - 2i$ is also a zero of the function.

When you are given all of the zeros of a polynomial function and asked to determine the function, use the zeros to write the factors and multiply them together. The result will be the polynomial function.

Example 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^3 + x^2 - 7x - 15$ and use them to sketch a rough graph.

Part A Find all of the zeros.

Step 1 Determine the total number of zeros.

Since $f(x)$ has degree 3, the function has 3 zeros.

(continued on the next page)

Step 2 Determine the type of zeros.

Examine the number of sign changes for $f(x)$ and $f(-x)$.

$$f(x) = \underbrace{x^3 + x^2}_{\text{no}} \underbrace{- 7x}_{\text{yes}} \underbrace{- 15}_{\text{no}} \qquad f(-x) = \underbrace{-x^3 + x^2}_{\text{yes}} \underbrace{+ 7x}_{\text{no}} \underbrace{- 15}_{\text{yes}}$$

Because there is 1 sign change for the coefficients of $f(x)$, the function has 1 positive real zero. Because there are 2 sign changes for the coefficients of $f(-x)$, $f(x)$ has 2 or 0 negative real zeros. Thus, $f(x)$ has 3 real zeros, or 1 real zero and 2 imaginary zeros.

Step 3 Determine the real zeros.

List some possible values, and then use synthetic substitution to evaluate $f(x)$ for real values of x.

x	1	1	−7	−15
−3	1	−2	−1	−12
−2	1	−1	−5	−5
−1	1	0	−7	−8
0	1	1	−7	−15
1	1	2	−5	−20
2	1	3	−1	−17
3	1	4	5	0
4	1	5	13	37

3 is a zero of the function, and the depressed polynomial is $x^2 + 4x + 5$. Since it is quadratic, use the Quadratic Formula. The zeros of $f(x) = x^2 + 4x + 5$ are $-2 - i$ and $-2 + i$.

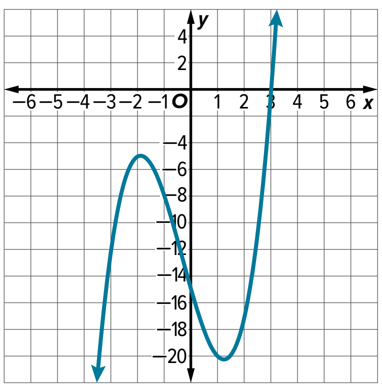
The function has zeros at 3, $-2 - i$ and $-2 + i$.

Part B Sketch a rough graph.

The function has one real zero at $x = 3$, so the function goes through (3, 0) and does not cross the x-axis at any other place.

Because the degree is odd and the leading coefficient is positive, the end behavior is that as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Use this information and points with coordinates found in the table above to sketch the graph.



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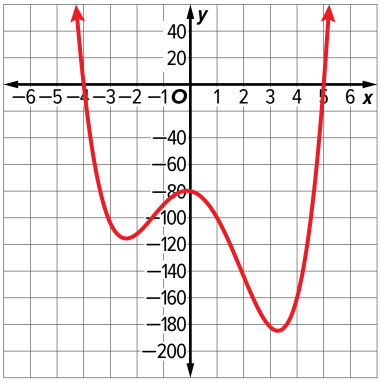
Check

Determine all of the zeros of $f(x) = x^4 - x^3 - 16x^2 - 4x - 80$, and use them to sketch a rough graph.

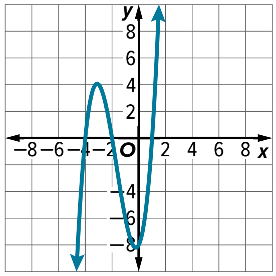
Real Zeros: -4, 5

Imaginary Zeros: $-2i$, $2i$



Example 4 Use a Graph to Write a Polynomial Function

Write a polynomial function that could be represented by the graph.



The graph crosses the x-axis 3 times, so the function is at least of degree 3. It crosses the x-axis at $x = -4$, $x = -2$, and $x = 1$, so its factors are $x + 4$, $x + 2$, and $x - 1$.

To determine a polynomial, find the product of the factors.

$y = (x + 4)(x + 2)(x - 1)$ Set the product of the factors equal to y.

$= (x^2 + 6x + 8)(x - 1)$ FOIL

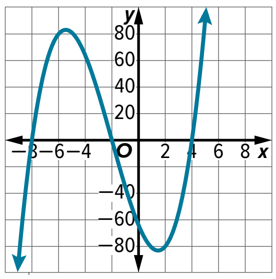
$= x^3 + 5x^2 + 2x - 8$ Multiply.

A polynomial that could be represented by the graph is

$y = x^3 + 5x^2 + 2x - 8$.

Check

Write a polynomial that could be represented by the graph. C



A. $y = x^3 - 6x^2 - 24x + 64$

B. $y = x^2 + 4x - 32$

C. $y = x^3 + 6x^2 - 24x - 64$

D. $y = x^3 - 64$

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Apply Example 5 Use Zeros to Graph a Polynomial Function

PROFIT MARGIN **A book publisher wants to release a special hardcover version of several Charles Dickens books. They know that if they charge \$5 or \$40, their profit will be \$0. Graph a polynomial function that could represent the company's profit in thousands of dollars given the price they charge for the book.**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: Let x represent the price that the publisher charges and let y represent the profit. I need to write and graph a polynomial function that relates x and y.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I know 5 and 40 are zeros of the function. I can use them to write factors to write an equation of the function.

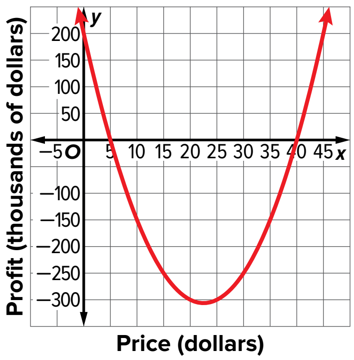
3 What is your solution?

Use your strategy to solve the problem.

What is a function that represents the given information?

$y = x^2 - 45x + 200$

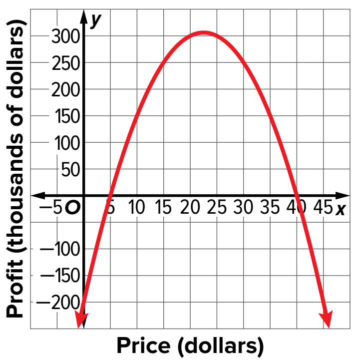
Graph the function.



Does this function make sense in the context of the situation? If not, explain why not and write and graph a more reasonable function.

No; the graph passed through the zeros, but did not show reasonable book prices that would result in profit.

$y = -x^2 + 45x - 200$



4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: With multiplying the function by -1, the new function shows that the profit is negative when charging less than \$5 per book. This makes sense in the context of the situation.

Problem-Solving Tip

Logical Reasoning When solving a problem it is important to use logical reasoning skills to analyze the problem.

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