The Remainder and Factor Theorems

Explore Remainders

Online Activity Use the interactive tool to complete the Explore.

INQUIRY How are the divisor and quotient of a polynomial related to its factors when the remainder is zero?

Today's Goals

- Evaluate functions by using synthetic substitution.
- Use the Factor Theorem to determine factors of polynomials.

Today's Vocabulary

synthetic substitution

depressed polynomial

Learn The Remainder Theorem

Polynomial division can be used to find the value of a function. From the Division Algorithm, we know that $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$ and that $f(x) = q(x) \cdot g(x) + r(x)$, where q and r are unique and the degree of r is less than the degree of g. Suppose we were to call the dividend $p(x)$ and the divisor $x - a$. Then the Division Algorithm would be $\frac{p(x)}{x-a} = q(x) + \frac{r}{x-a}$ and $p(x) = q(x) \cdot (x - a) + r$, where a is a constant and r is the remainder. Since any polynomial can be written in this form, evaluating $p(x)$ at a gives the following.

$p(x) = q(x) \cdot (x - a) + r$	Polynomial function $p(x)$
$p(a) = q(a) \cdot (a - a) + r$	Substitute a for x.
$p(a) = q(a) \cdot (0) + r$	$a - a = 0$
$p(a) = r$	$q(a) \cdot (0) = 0$

This shows how the Remainder Theorem can be used to evaluate a polynomial at $p(a)$.

Key Concept • Remainder Theorem

Words: For a polynomial $p(x)$ and a number a, the remainder upon division by $x - a$ is $p(a)$.

Example: Evaluate $p(x) = x^2 - 4x + 7$ when $x = 5$.

Synthetic division

$$\begin{array}{r|rrr} 5 & 1 & -4 & 7 \\ & & 5 & 5 \\ \hline & 1 & 1 & 12 \end{array}$$

$p(5) = 12$

Direct substitution

$p(x) = x^2 - 4x + 7$

$p(5) = 5^2 - 4(5) + 7$

$p(5) = 12$

Your Notes

Applying the Remainder Theorem to evaluate a function is called **synthetic substitution**. You may find that synthetic substitution is a more convenient way to evaluate a polynomial function, especially when the degree of the function is greater than 2.

Study Tip

Missing terms Remember to include zeros as placeholders for any missing terms in the polynomial.

Example 1 Synthetic Substitution

Use synthetic substitution to find $f(-3)$ if $f(x) = -2x^4 + 3x^2 - 15x + 9$.

By the Remainder Theorem, $f(-3)$ is the remainder of $\frac{f(x)}{x-(-3)}$.

−3	−2	0	3	−15	9
		6	−18	45	−90
	−2	6	−15	30	−81

The remainder is −81. Therefore, $f(-3) =$ −81.

Use direct substitution to check.

$f(x) = -2x^4 + 3x^2 - 15x + 9$ Original function

$f(-3) = -2(-3)^4 + 3(-3)^2 - 15(-3) + 9$ Substitute −3 for x.

$= -162 + 27 + 45 + 9$ or -81 True

Check

Use synthetic substitution to evaluate $f(x) = -6x^3 + 52x^2 - 27x - 31$.

$f(8) =$ 9

Think About It!

How could you use the function and synthetic substitution to estimate the number of eggs produced in 1990? What assumption would you have to make to solve this problem?

Sample answer: Because 1990 is 10 years before 2000, use synthetic substitution to find the value of $f(-10)$. I would have to assume that the production of eggs followed the same model before 2000.

Example 2 Apply the Remainder Theorem

EGG PRODUCTION **The total production of eggs in billions in the United States can be modeled by the function $f(x) = 0.007x^3 - 0.149x^2 + 1.534x + 84.755$, where x is the number of years since 2000. Predict the total production of eggs in 2025.**

Since 2025 − 2000 = 25, use synthetic substitution to determine $f(25)$.

25	0.007	−0.149	1.534	84.755
		0.175	0.65	54.6
	0.007	0.026	2.184	139.355

In 2025, approximately 139.355 billion eggs will be produced in the United States.

Go Online You can complete an Extra Example online.

Check

KITTENS The ideal weight of a kitten in pounds is modeled by the function $f(x) = 0.009x^2 + 0.127x + 0.377$, where x is the age of the kitten in weeks. Determine the ideal weight of a 9-week-old kitten. Round to the nearest tenth.

2.3 pounds

Learn The Factor Theorem

When a binomial evenly divides a polynomial, the binomial is a factor of the polynomial. The quotient of this division is called a depressed polynomial. The **depressed polynomial** has a degree that is one less than the original polynomial.

A special case of the Remainder Theorem is called the Factor Theorem.

Key Concept • Factor Theorem

Words: The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Example:

$$\overbrace{x^3 - x^2 - 30x + 72}^{\text{dividend}} = \overbrace{(x^2 - 7x + 12)}^{\text{quotient}} \cdot \overbrace{(x + 6)}^{\text{divisor}} + \overbrace{0}^{\text{remainder}}$$

$x + 6$ is a factor of $x^3 - x^2 - 30x + 72$.

Go Online You may want to complete the Concept Check to check your understanding.

Example 3 Use the Factor Theorem

Show that $x + 8$ is a factor of $2x^3 + 15x^2 - 11x - 24$. Then find the remaining factors of the polynomial.

−8	2	15	−11	−24
		−16	8	24
	2	−1	−3	0

Because the remainder is 0, $x + 8$ is a factor of the polynomial by the Factor Theorem. So $2x^3 + 15x^2 - 11x - 24$ can be factored as $(x + 8)(2x^2 - x - 3)$. The depressed polynomial is $2x^2 - x - 3$.

Check to see if this polynomial can be factored.

$2x^2 - x - 3 = (2x - 3)(x + 1)$ Factor the trinomial.

Therefore, $2x^3 + 15x^2 - 11x - 24 = (x + 8)(2x - 3)(x + 1)$.

Study Tip

Factoring Some depressed polynomials may not be factorable. In that case, the only factors are the divisor and the depressed polynomial.

Talk About It

Suppose you were asked to determine whether $3x + 4$ is a factor of $3x^3 - 2x^2 - 8x$. Describe the steps necessary to find a solution.

Sample answer: First, divide the factor and the polynomial by 3 so that the x-term of the factor is in the form $x - a$. In this case, $a = -\frac{4}{3}$. Then, use synthetic division to determine the remainder. If the remainder is 0, then $3x + 4$ is a factor.

Check

Select all of the factors of $3x^3 + 10x^2 - 27x - 10$. A, B, E

A. $x - 2$

B. $x + 5$

C. $x + 9$

D. $x - 10$

E. $3x + 1$

F. $3x - 10$

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

Record your observations here.

See students' observations.

Go Online You can complete an Extra Example online.