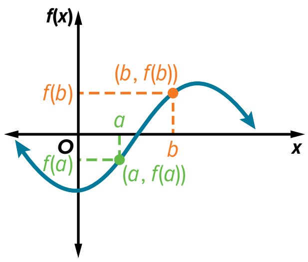
Analyzing Graphs of Polynomial Functions

Learn The Location Principle

If the value of $f(x)$ changes signs from one value of x to the next, then there is a zero between those two x-values. This is called the Location Principle.

Key Concept • Location Principle

Suppose $y = f(x)$ represents a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.



Example 1 Locate Zeros of a Function

Determine the consecutive integer values of x between which each real zero of $f(x) = x^4 - 2x^3 - x^2 + 1$ is located. Then draw the graph.

Step 1 Make a table.

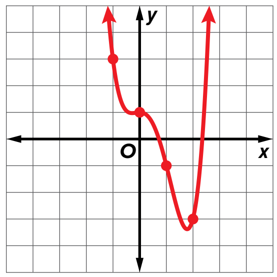
Because $f(x)$ is a fourth-degree polynomial, it will have as many as 4 real zeros or none at all.

x	−2	−1	0	1	2	3	4
$f(x)$	29	3	1	−1	−3	19	113

Using the Location Principle, there are zeros between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$.

Step 2 Sketch the graph.

Use the table to sketch the graph and find the locations of the zeros.



Check

Use technology to check the location of the zeros.

Input the function into a graphing calculator to confirm that the function crosses the x-axis between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$.

You can find more accurate values of the zeros by using the **zero** feature in the CALC menu to find $x \approx 0.7213$ and $x \approx 2.3486$, which confirms the estimates.

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Today's Goals

- Approximate zeros by graphing polynomial functions.
- Find extrema of polynomial functions.

Think About It!

Not all real zeros can be found by using the Location Principle. Provide an example where $f(a) > 0$ and $f(b) > 0$, but there is a zero between $x = a$ and $x = b$.

Sample answer: For $f(x) = x^2$, $f(-1) = 1$ and $f(1) = 1$, but there is a zero at $x = 0$.

Think About It!

How can you adjust the table on your graphing calculator to give a more precise interval for the value of each zero?

Sample answer: You can adjust the table setting so make the interval between each x-value smaller.

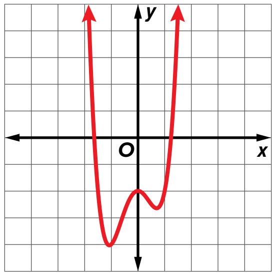
Your Notes

Check

Determine the consecutive integer values of x between which each real zero of $f(x) = 2x^4 + x^3 - 3x^2 - 2$ is located. Then draw the graph.

$x =$ __−2__ and $x =$ __−1__

$x =$ __1__ and $x =$ __2__



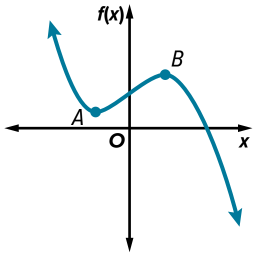
Study Tip

Turning Points Relative maxima and relative minima of a function are sometimes called turning points.

Learn Extrema of Polynomials

Extrema occur at relative maxima or minima of the function.

Point A is a relative minimum, and point B is a relative maximum. Both points A and B are extrema. The graph of a polynomial of degree n has at most $n - 1$ extrema.

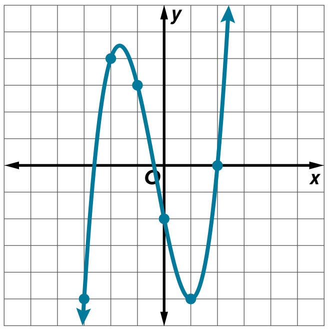


Example 2 Identify Extrema

Use a table to graph $f(x) = x^3 + x^2 - 5x - 2$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

Step 1 Make a table of values and graph the function.

x	$f(x)$
−4	−30
−3	−5
−2	4
−1	3
0	−2
1	−5
2	0
3	19



Step 2 Estimate the locations of the extrema.

The value of $f(x)$ at $x = -2$ is greater than the surrounding points indicating a maximum *near* $x =$ __−2__.

The value of $f(x)$ at $x = 1$ is less than the surrounding points indicating a minimum *near* $x =$ __1__.

You can use a graphing calculator to find the extrema of a function and confirm your estimates.

Study Tip

Extrema When graphing with a calculator, keep in mind that a polynomial of degree n has at most $n - 1$ extrema. This will help you to determine whether your viewing window is allowing you to see all of the extrema of the graph.

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Check

Use a table of values of $f(x) = -x^4 - x^3 + 5x^2 + x - 3$ to estimate the x-coordinates at which the relative maxima and relative minima occur.

x	$f(x)$
−3	**−15**
−2	**7**
−1	**1**
0	**−3**
1	**1**
2	**−5**
3	**−63**

The relative maxima occur near $x =$ **−2** and $x =$ **1**.

The relative minimum occurs near $x =$ **0**.

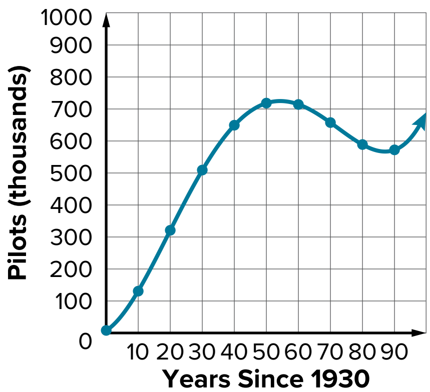
Example 3 Analyze a Polynomial Function

PILOTS **The total number of certified pilots in the United States is approximated by $f(x) = 0.0000903x^4 - 0.0166x^3 + 0.762x^2 + 6.317x + 7.708$, where x is the number of years after 1930 and $f(x)$ is the number of pilots in thousands. Graph the function and describe its key features over the relevant domain.**

Step 1 Graph the function.

Make a table of values. Plot the points and connect them with a smooth curve.

x	$f(x)$
0	7.708
10	131.381
20	320.496
30	507.961
40	648.356
50	717.933
60	714.616
70	658.001
80	589.356
90	571.621



Step 2 Describe the key features.

Domain and Range

The domain and range of the function is all real numbers. Because the function models years after 1930, the relevant domain and range are $\{x \mid x \geq$ **0** $\}$ and $\{f(x) \mid f(x) \geq$ **7.708** $\}$.

(continued on the next page)

Go Online

You can learn how to graph and analyze a polynomial function on a graphing calculator by watching the video online.

Think About It!

What trends in the number of pilots does the graph suggest?

Sample answer: The number of pilots peaked around 1985 and declined to a relative minimum around 2015. After 2015, the number of pilots will continue to increase.

Think About It!

It is reasonable that the trend will continue indefinitely? Explain

No; sample answer: It is unreasonable to assume that the number of pilots will continue to increase indefinitely. At some point, the number will stay relatively constant or begin to decline.

Study Tip

Assumptions Determining the end behavior for the graph of a polynomial that models data assumes that the trend continues and there are no other relative maxima or minima.

Extrema

There is a relative maximum between 1980 and 1990 and a relative minimum between 2010 and 2020 in the relevant domain.

End Behavior

As $x \to \infty$, $f(x) \to \infty$.

Intercepts

In the relevant domain, the y-intercept is at (0, 7.708). There is no x-intercept, or zero, because the function begins at a value greater than 0 and as $x \to \infty$, $f(x) \to \infty$.

Symmetry The graph of the function does not have symmetry.

Check

COINS The number of quarters produced by the United States Mint can be approximated by the function $f(x) = 16.4x^3 - 149.5x^2 - 148.9x + 3215.4$, where x is the number of years since 2005 and $f(x)$ is the total number of quarters produced in millions. Use the graph of the function to complete the table and describe its key features.

Part A Complete the table.

x, Years	$f(x)$, Quarters (millions)
0	3215
2	2451
4	1277
6	482
8	853
10	3176

Part B Describe the key features.

The relevant domain is $\{x \mid x \geq 0\}$.

The relevant range is $\{f(x) \mid f(x) \geq \approx 482\}$.

There is a relative minimum between 2011 and 2013.

The y-intercept is 3215.

The graph of the function does not have symmetry.

It is unreasonable to assume that the trend will continue indefinitely.

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Example 4 Use a Polynomial Function and Technology to Model

BACKPACKS **The table shows U.S. backpack sales in millions of dollars, according to the Travel Goods Association. Make a scatter plot and a curve of best fit to show the trend over time. Then determine the backpack sales in 2015.**

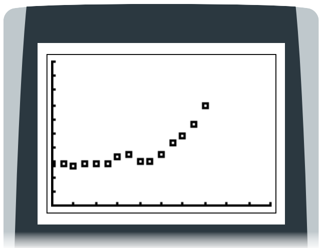
Year	Sales (million $)	Year	Sales (million $)
2000	1140	2008	1246
2001	1144	2009	1235
2002	1113	2010	1419
2003	1134	2011	1773
2004	1164	2012	1930
2005	1180	2013	2255
2006	1364	2014	2779
2007	1436		

Step 1 Enter the data.

Let the year 2000 be represented by 0. Enter the years since 2000 in List 1. Enter the backpack sales in List 2.

Step 2 Graph the scatter plot.

Choose the scatter plot feature in the **STAT PLOT** menu. Use List 1 for the **Xlist** and List 2 for the **Ylist**. Change the viewing window so that all the data are visible.



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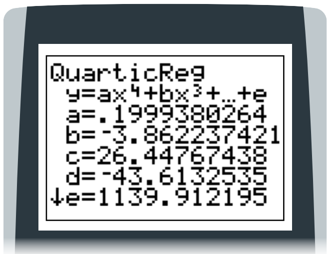
Step 3 Determine the polynomial function of best fit.

To determine the model that best fits the data, perform linear, quadratic, cubic, and quartic regressions, and compare the coefficients of determination, r^2. The polynomial with a coefficient of determination closest to 1 will fit the data best.



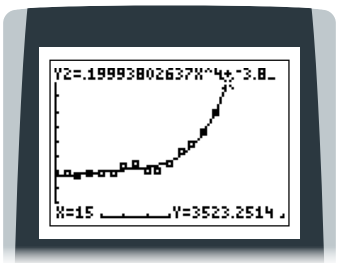
A quartic function fits the data best.

The regression equation with coefficients rounded to the nearest tenths is:

$$y \approx 0.2x^4 - 3.9x^3 + 26.4x^2 - 43.6x + 1139.9.$$

Step 4 Graph and evaluate the regression function.

Assuming that the trend continues, the graph of the function can be used to predict backpack sales for a specific year. To determine the total sales in 2015, find the value of the function for $x =$ 15.



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In 2015, there were about $ 3.523 billion in backpack sales.

Math History Minute

By the age of 20, Italian mathematician **Maria Gaetana Agnesi (1718–1799)** had started working on her book *Analytical Institutions,* which was published in 1748. Early chapters included problems on maxima, minima, and turning points. Also described was a cubic curve called the "witch of Agnesi," which was translated incorrectly from the original Italian.

Think About It!

Explain the approximation that is made when using the model to determine the backpack sales in a specific year.

Sample answer: The model is an approximation of the entire set of data, but it may not be accurate at a specific year. Also, the model may not be a good representation for specific years outside of the domain of the original data because it was not considered when creating the model.

Check

TREES To estimate the amount of lumber that can be harvested from a tree, foresters measure the diameter of each tree. Determine the polynomial function of best fit, where x represents the diameter of a tree in inches and y is the estimated volume measured in board feet. Then estimate the volume of a tree with a diameter of 35 inches.

Diam (in.)	17	19	20	23	25	28	32	38	39	41
Vol (100s of board ft)	19	25	32	57	71	113	123	252	259	294

Polynomial function of best fit:
$y =$ −0.0006 $x^4 +$ 0.08 $x^3 -$ 3.55 $x^2 +$ 69.11 $x -$ 491.39

The estimated volume of a 35-inch diameter tree to the nearest board foot is 188 of 100s board ft.

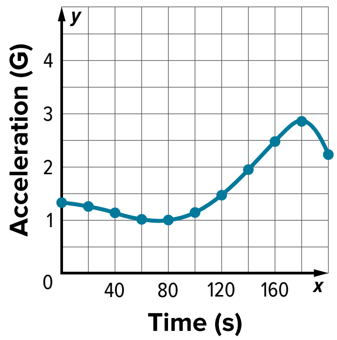
Example 5 Find Average Rate of Change

ROCKETS **The Ares-V rocket was designed to carry as much as 75 tons of supplies and 4 astronauts to the Moon and possibly even to Mars. The table shows the expected g-force on the rocket over the course of its 200-second launch.**

Time (s)	Acceleration (Gs)	Time (s)	Acceleration (Gs)
0	1.34	120	1.46
20	1.26	140	1.93
40	1.12	160	2.47
60	1.01	180	2.84
80	1	200	2.2
100	1.15		

Part A Find the average rate of change.

Sketch the graph, and estimate the average rate of change of the acceleration. Then check your results algebraically.



Estimate: From the graph, the change in the y-values is about 0.9, and the change in the x-values is 200. So, the rate of change is about $\frac{0.9}{200}$ or 0.0045.

Check algebraically:

The average rate of change is
$\frac{f(200) - f(0)}{200 - 0} = \frac{2.2 - 1.34}{200 - 0}$ or 0.0043.

Part B Interpret the results.

From 0 to 200 seconds, the average rate of change in acceleration was an increase of 0.0043 Gs per second.

Go Online You can complete an Extra Example online.

Study Tip

g-force One G is the acceleration due to gravity at the Earth's surface. Defined as 9.80665 meters per second squared, this is the g-force you experience when you stand still on Earth. On a roller coaster, you experience 0 Gs and feel weightless at the top of the hills, and you can experience a g-force of 6 Gs or more as you are pushed into your seat at the bottom of the hills.

Think About It!

Does the average rate of change from 0 to 200 seconds accurately describe the acceleration of the launch? Justify your reasoning.

No; sample answer: For the first 80 seconds, the acceleration is relatively slow. For the next 100 seconds, it increases rapidly. Then, it decreases. The average rate of change does not describe the changes in acceleration over time.