Solving Quadratic Equations by Completing the Square

Learn Solving Quadratic Equations by Using the Square Root Property

You can use square roots to solve equations like $x^2 - 49 = 0$. Remember that 7 and -7 are both square roots of 49 because $7^2 = 49$ and $(-7)^2 = 49$. Therefore, the solution set of $x^2 - 49 = 0$ is $\{-7, 7\}$. This can be written as $\{\pm 7\}$.

Key Concept • Square Root Property

Words: To solve a quadratic equation in the form $x^2 = n$, take the square root of each side.

Symbols: For any number $n \geq 0$, if $x^2 = n$, then $x = \pm\sqrt{n}$.

Example: $x^2 = 121$, $x = \pm\sqrt{121}$ or ± 11

Not all quadratic equations have solutions that are whole numbers. Roots that are irrational numbers may be written as exact answers in radical form or as approximate answers in decimal form when a calculator is used. Sometimes solutions of quadratic equations are not real numbers. Solutions that are complex numbers can be written in the form $a + bi$, where $b \neq 0$.

Example 1 Solve a Quadratic Equation with Rational Roots

Solve $x^2 - 4x + 4 = 25$ by using the Square Root Property.

$x^2 - 4x + 4 = 25$	Original equation
$(x - 2)^2 = 25$	Factor.
$x - 2 = \pm\sqrt{25}$	Square Root Property
$x - 2 = \pm$ 5	$25 = 5(5)$ or $-5(-5)$
$x = 2 \pm 5$	Add 2 to each side.
$x = 2 + 5$ or $x = 2 -$ 5	Write as two equations.
$x =$ 7 $\quad x = -3$	Simplify.

The solution set is $\{x \mid x =$ −3, 7$\}$.

Check

Solve $x^2 - 38x + 361 = 576$ by using the Square Root Property.

$x =$ −5 and 43

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Today's Goal

- Solve quadratic equations by using the Square Root Property.
- Complete the square in quadratic expressions to solve quadratic equations.
- Complete the square in a quadratic function to interpret key features of its graph.

Today's Vocabulary

completing the square

vertex form

projectile motion problems

Think About It!

How can you determine whether an equation of the form $x^2 = n$ will have an answer that is a whole number?

Sample answer: If n is a perfect square, $x^2 = n$ will have an answer that is a whole number.

Study Tip

When using the Square Root Property, remember to include the ± before the radical.

Go Online An alternate method is available for this example.

Your Notes

Example 2 Solve a Quadratic Equation with Irrational Roots

Solve $x^2 + 24x + 144 = 192$ by using the Square Root Property.

$x^2 + 24x + 144 = 192$	Original equation
$(x + 12)^2 = 192$	Factor.
$x + 12 = \pm\sqrt{192}$	Square Root Property
$x + 12 = \pm 8\sqrt{3}$	$\sqrt{192} = 8\sqrt{3}$
$x = \underline{-12} \pm 8\sqrt{3}$	Subtract 12 from each side.
$x = -12 + 8\sqrt{3}$ or $\underline{-12} - 8\sqrt{3}$	Write as two equations.
$x \approx \underline{1.86}, \underline{-25.86}$	Use a calculator.

The exact solutions are $-12 - 8\sqrt{3}$ and $-12 + 8\sqrt{3}$. The approximate solutions are -25.86 and 1.86.

Watch Out!

Perfect Squares The constant, 192, on the right side of the equation is not a perfect square. This means that the roots will be irrational numbers.

Example 3 Solve a Quadratic Equation with Complex Solutions

Solve $2x^2 - 92x + 1058 = -72$ by using the Square Root Property.

$2x^2 - 92x + 1058 = -72$	Original equation
$x^2 - 46x + 529 = \underline{-36}$	Divide each side by 2.
$(x - \underline{23})^2 = -36$	Factor.
$x - 23 = \pm\sqrt{-36}$	Square Root Property
$x - 23 = \pm \underline{6i}$	$\sqrt{-36} = 6i$
$x = \underline{23} \pm 6i$	Add 23 to each side.
$x = 23 + 6i$ or $\underline{23} - 6i$	Write as two equations.

The solutions are $23 + 6i$ and $23 - 6i$.

Think About It!

Can you solve a quadratic equation by completing the square if the coefficient of the x^2-term is not 1? Justify your argument.

Yes; sample answer: first divide the equation by the coefficient.

Explore Using Algebra Tiles to Complete the Square

Online Activity Use algebra tiles to complete the Explore.

INQUIRY How does forming a square to create a perfect square trinomial help you solve quadratic equations?

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Learn Solving Quadratic Equations by Completing the Square

All quadratic equations can be solved by using the properties of equality to manipulate the equation until one side is a perfect square. This process is called **completing the square**.

Key Concept • Completing the Square

Words: To complete the square for any quadratic expression of the form $x^2 + bx$, follow the steps below.

Step 1 Find one half of b, the coefficient of x.
Step 2 Square the result in Step 1.
Step 3 Add the result of Step 2 to $x^2 + bx$.

Symbols: $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

To solve an equation of the form $x^2 + bx + c = 0$ by completing the square, first subtract c from each side of the equation. Then add $\left(\frac{b}{2}\right)^2$ to each side of the equation and solve for x.

Example 4 Complete the Square

Find the value of c that makes $x^2 - 7x + c$ a perfect square. Then write the expression as a perfect square trinomial.

Step 1 Find one half of -7. $\frac{-7}{2} =$ -3.5

Step 2 Square the result from Step 1. $(-3.5)^2 =$ 12.25

Step 3 Add the result from Step 2 to $x^2 - 7x$. $x^2 - 7x + 12.25$

The expression $x^2 - 7x + 12.25$ can be written as $(x - 3.5)^2$.

Example 5 Solve by Completing the Square

Solve $x^2 + 18x - 4 = 0$ by completing the square.

$x^2 + 18x - 4 = 0$	Original equation
$x^2 + 18x = 4$	Add 4 to each side.
$x^2 + 18x + 81 = 4 + 81$	Add $\left(\frac{b}{2}\right)^2$ to each side.
$(x + 9)^2 = 85$	Factor.
$x + 9 = \pm\sqrt{85}$	Square Root Property
$x = -9 \pm \sqrt{85}$	Subtract 9 from each side.
$x = -9 + \sqrt{85}$ or	Write as two equations.
$x = -9 - \sqrt{85}$	
$x \approx 0.22$, or -18.22	Simplify.

The solution set is $\left\{x \mid x = -9 - \sqrt{85}, -9 + \sqrt{85}\right\}$.

Go Online to see Example 6.

Talk About It

If a and b are real numbers, can the value of c ever be negative? Explain your reasoning.

No; sample answer: because this value is equal to half of b squared, and the square of a real number is never negative.

Think About It!

Why do we first add 4 to each side?

Sample answer: In order to complete the square, the expression on one side of the equation must be of the form $x^2 + bx$.

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Example 7 Solve When a is Not 1

Solve $4x^2 - 12x - 27 = 0$ by completing the square.

$4x^2 - 12x - 27 = 0$	Original equation
$x^2 - \underline{3}\,x - \frac{27}{4} = 0$	Divide each side by 4.
$x^2 - 3x = \underline{\frac{27}{4}}$	Add $\frac{27}{4}$ to each side.
$x^2 - 3x + \underline{\frac{9}{4}} = \frac{27}{4} + \underline{\frac{9}{4}}$	Add $\left(\frac{b}{2}\right)^2$ or $\frac{9}{4}$ to each side.
$\left(x - \underline{\frac{3}{2}}\right)^2 = \underline{9}$	Factor.
$x - \frac{3}{2} = \pm\,\underline{3}$	Square Root Property
$x = \underline{\frac{3}{2}} \pm 3$	Add $\frac{3}{2}$ to each side.
$x = \frac{3}{2} + 3$ or $x = \frac{3}{2} - \underline{3}$	Write as two equations.
$x = \underline{\frac{9}{2}}$ $\quad x = \underline{-\frac{3}{2}}$	Simplify.

The solution set is $\left\{x \,\middle|\, x = -\frac{3}{2}, \frac{9}{2}\right\}$.

Check

Solve $6x^2 - 21x + 9 = 0$ by completing the square.

$x =$ 3, 0.5

Think About It!

Compare and contrast the solutions of this equation and the ones in the previous example. Explain.

Sample answer: Both equations have two solutions. This equation has imaginary solutions because I took the square root of a negative number, and the previous equation had real solutions because I took the square root of a positive number.

Example 8 Solve Equations with Imaginary Solutions

Solve $3x^2 - 72x + 465 = 0$ by completing the square.

$3x^2 - 72x + 465 = 0$	Original equation
$x^2 - \underline{24}\,x + \underline{155} = 0$	Divide each side by 3.
$x^2 - 24x = \underline{-155}$	Subtract 155 from each side.
$x^2 - 24x + \underline{144} = -155 + \underline{144}$	Add $\left(\frac{b}{2}\right)^2$ to each side.
$(x - \underline{12})^2 = \underline{-11}$	Factor.
$x - 12 = \pm\sqrt{-11}$	Square Root Property
$x - 12 = \pm i\sqrt{11}$	$\sqrt{-1} = i$
$x = \underline{12} \pm i\sqrt{11}$	Add 12 to each side.
$x = 12 + i\sqrt{11}$ or $12 - i\sqrt{11}$	Write as two equations.

The solution set is $\left\{x \,\middle|\, x = 12 + i\sqrt{11}, 12 - i\sqrt{11}\right\}$.

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Learn Quadratic Functions in Vertex Form

When a function is given in standard form, $y = ax^2 + bx + c$, you can complete the square to write it in vertex form.

Key Concept • Vertex Form of a Quadratic Function	
Words: The vertex form of a quadratic function is $y = a(x - h)^2 + k$.	
Symbols: Standard Form $y = ax^2 + bx + c$	Vertex Form $y = a(x - h)^2 + k$ The vertex is (h, k).
Example: Standard Form $y = 2x^2 + 12x + 16$	Vertex Form $y = 2(x + 3)^2 - 2$ The vertex is $(-3, -2)$.

After completing the square and writing a quadratic function in vertex form, you can analyze key features of the function. The vertex is (h, k) and $x = h$ is the equation of the axis of symmetry. The shape of the parabola and the direction that it opens are determined by a. The value of k is a minimum value if $a > 0$ or a maximum value if $a < 0$.

The path that an object travels when influenced by gravity is called a *trajectory*, and trajectories can be modeled by quadratic functions. The formula below relates the height of the object $h(t)$ and time t, where g is acceleration due to gravity, v is the initial velocity of the object, and h_0 is the initial height of the object.

$$h(t) = -\frac{1}{2}gt^2 + vt + h_0$$

The acceleration due to gravity g is 9.8 meters per second squared or 32 feet per second squared. Problems that involve objects being thrown or dropped are called **projectile motion problems**.

Think About It!

What is the minimum value of $y = 2(x - 3)^2 - 1$? How do you know that this value is a minimum?

−1; Sample answer: Because a is positive, the graph has a minimum, and the vertex is (3, −1).

Example 9 Write Functions in Vertex Form

Write $y = -x^2 - 12x - 9$ in vertex form.

$y = -x^2 - 12x - 9$	Original function
$y = (-x^2 - 12x) - 9$	Group $ax^2 + bx$.
$y = -(x^2 + 12x) - 9$	Factor out −1.
$y = -(x^2 + 12x + \underline{36}) - 9 - (\underline{-1})(\underline{36})$	Complete the square.
$y = -(x + \underline{6})^2 + \underline{27}$	Simplify.

Watch Out!

The coefficient of the x^2-term must be 1 before you can complete the square.

Check

Write each function in vertex form.

a. $y = x^2 + 8x - 3$

$y = (x \underline{+4})^2 \underline{-19}$

b. $y = -3x^2 - 6x - 5$

$y = \underline{-3}(x \underline{+1})^2 \underline{-2}$

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Example 10 Determine the Vertex and Axis of Symmetry

Consider $y = 3x^2 - 12x + 5$.

Part A Write the function in vertex form.

$y = 3x^2 - 12x + 5$	Original equation
$y = (3x^2 - 12x) + 5$	Group $ax^2 + bx$.
$y = 3(x^2 - 4x) + 5$	Factor.
$y = 3(x^2 - 4x + 4) + 5 - 3(4)$	Complete the square.
$y = 3(x - 2)^2 - 7$	Simplify.

Part B Find the axis of symmetry.

The axis of symmetry is $x = h$ or $x =$ 2.

Part C Find the vertex, and determine if it is a maximum or minimum.

The vertex is (h, k) or (2, −7). Because $a > 0$, this is a minimum.

Think About It!

How would your equation for the axis of symmetry change if the vertex form of the equation was $y = 3(x + 2)^2 - 7$? Justify your argument.

Sample answer: The axis of symmetry would be $x = -2$ because the vertex form of the function can be rewritten as $y = [x - (-2)]^2 - 7$.

Example 11 Model with a Quadratic Function

FIREWORKS **If a firework is launched 1 foot off the ground at a velocity of 128 feet per second, write a function for the situation. Then find and interpret the axis of symmetry and vertex.**

Step 1 Write the function.

$h(t) = -\frac{1}{2}gt^2 + vt + h_0$	Function for projectile motion
$h(t) = -\frac{1}{2}(32)t^2 + 128t + 1$	$g = 32\ \frac{\text{ft}}{\text{s}^2}$, $v = 128\ \frac{\text{ft}}{\text{s}}$, $h_0 = 1$ ft
$h(t) = -16t^2 + 128t + 1$	Simplify.

Step 2 Rewrite the function in vertex form.

$h(t) = (-16t^2 + 128t) + 1$	Group $ax^2 + bx$.
$h(t) = -16(t^2 - 8t) + 1$	Factor.
$h(t) = -16(t^2 - 8t + 16) + 1 - 16(-16)$	Complete the square.
$h(t) = -16(t - 4)^2 + 257$	Simplify.

Step 3 Find and interpret the axis of symmetry.

Because the axis of symmetry divides the function into two equal halves, the firework will be at the same height after 2 seconds as it is after 6 seconds.

Step 4 Find and interpret the vertex.

The vertex is the maximum of the function because $a < 1$. So the firework reached a maximum height of 257 feet after 4 seconds.

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Think About It!

If the firework reaches a height of 241 feet after 3 seconds, what is the height of the firework after 5 seconds? Justify your answer.

241 feet; Sample answer: Because the axis of symmetry is $t = 4$, the height of the firework will be the same after 3 seconds and after 5 seconds.

Study Tip

Vertex When you interpret the vertex of a function, it is important to also consider the value of a when the function is in vertex or standard form. The value of a will tell you whether the vertex is a maximum or minimum.