Complex Numbers

Learn Pure Imaginary Numbers

In your math studies so far, you have worked with real numbers. However, some equations such as $x^2 + x + 1 = 0$ do not have real solutions. This led mathematicians to define imaginary numbers. The **imaginary unit *i*** is the principal square root of -1. Thus, $i = \sqrt{-1}$ and $i^2 = -1$.

Numbers of the form $6i$, $-2i$, and $i\sqrt{3}$ are called pure imaginary numbers. A **pure imaginary number** is a number of the form bi, where b is a real number and $i = \sqrt{-1}$. For any positive real number $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$ or bi.

The Commutative and Associative Properties of Multiplication hold true for pure imaginary numbers. The first few powers of i are shown.

$i^1 = i$ $\qquad$ $i^2 = -1$ $\qquad$ $i^3 = i^2 \cdot i$ or $-i$

Example 1 Square Roots of Negative Numbers

Simplify $\sqrt{-294}$.

$\sqrt{-294} = \sqrt{-1 \cdot 7^2 \cdot 6}$ — Factor the radicand.

$= \sqrt{-1} \cdot \sqrt{7^2} \cdot \sqrt{6}$ — Factor out the imaginary unit.

$= i \cdot 7 \cdot \sqrt{6}$ or $7i\sqrt{6}$ — Simplify.

Check

Simplify $\sqrt{-75}$. C

A $i\sqrt{75}$

B $3i\sqrt{5}$

C $5i\sqrt{3}$

D $-3\sqrt{5}$

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Today's Goals

- Perform operations with pure imaginary numbers.
- Perform operations with complex numbers.

Today's Vocabulary

imaginary unit i

pure imaginary number

complex number

complex conjugates

rationalizing the denominator

Go Online You may want to complete the Concept Check to check your understanding.

Study Tip

Square Factors When factoring an expression under a radical, look for perfect square factors.

Your Notes

Talk About It

How can an expression with two imaginary expressions, $\sqrt{-10}$ and $\sqrt{-15}$, have a product that is real?

Sample answer: Because $\sqrt{-10}$ and $\sqrt{-15}$ are both imaginary, you can factor out the imaginary unit i from each. The product of $i \cdot i = i^2 = (\sqrt{-1})^2 = -1$. So, the product of two imaginary numbers is a negative real number.

Think About It!

Compare and contrast the methods.

Sample answer: Both methods solve for x and yield two solutions. For the second method, though, the equation is factored rather than simplified by using the Square Root Property.

Example 2 Products of Pure Imaginary Numbers

Simplify $\sqrt{-10} \cdot \sqrt{-15}$.

$\sqrt{-10} \cdot \sqrt{-15} = i\sqrt{10} \cdot i\sqrt{15}$	$i = \sqrt{-1}$
$= i^2 \cdot \sqrt{150}$	Multiply.
$= -1 \cdot \sqrt{25} \cdot \sqrt{6}$	Simplify.
$= -5\sqrt{6}$	Multiply.

Check

Simplify $\sqrt{-16} \cdot \sqrt{-25}$.

-20

Example 3 Equation with Pure Imaginary Solutions

Solve $x^2 + 81 = 0$.

$x^2 + 81 = 0$	Original equation
$x^2 = -81$	Subtract 81 from each side.
$x = \pm \sqrt{-81}$	Square Root Property
$x = \pm 9i$	Simplify.

ALTERNATE METHOD

$x^2 + 81 = 0$	Original equation
$x^2 + 9^2 = 0$	$81 = 9^2$
$x^2 - (-9)^2 = 0$	Rewrite in the difference of squares pattern.
$(x + 9i)(x - 9i) = 0$	Difference of squares: $\sqrt{-9^2} = \sqrt{-81} = 9i$
$(x + 9i) = 0$ or $(x - 9i) = 0$	Zero Product Property
$x = -9i$ $\quad$ $x = 9i$	Solve.

Check

Solve $3x^2 + 27 = 0$.

$x = 3i$, $x = -3i$

Explore Factoring Prime Polynomials

Online Activity Use guiding exercises to complete the Explore.

INQUIRY Can you factor a prime polynomial?

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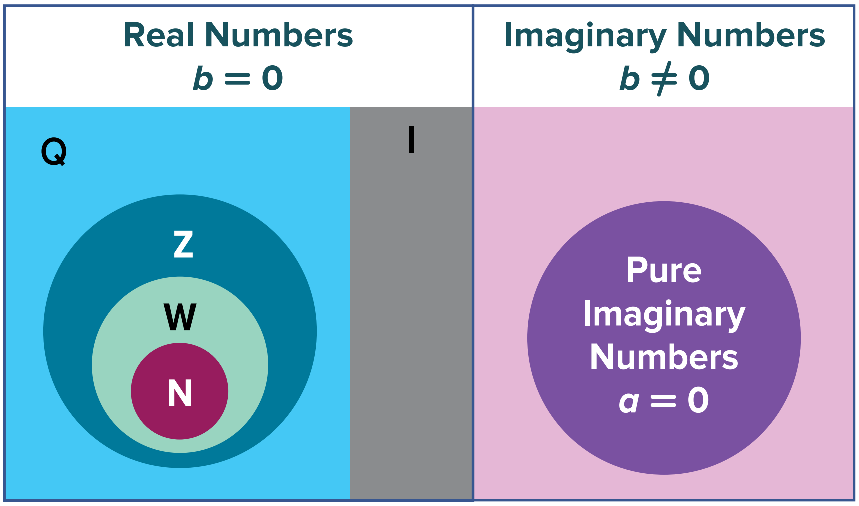
Learn Complex Numbers

Key Concept • Complex Numbers

A **complex number** is any number that can be written in the form $a + bi$, where a and b are real numbers and $\boldsymbol{i}$ is the imaginary unit. a is called the real part, and b is called the imaginary part.

The Venn diagram shows the set of complex numbers. Notice that all of the real numbers are part of the set of complex numbers.

Complex Numbers ($a + bi$)



Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. The Commutative and Associative Properties of Multiplication and Addition and the Distributive Property hold true for complex numbers. To add or subtract complex numbers, combine like terms. That is, combine the real parts, and combine the imaginary parts.

Two complex numbers of the form $a + bi$ and $a - bi$ are called **complex conjugates**. The product of complex conjugates is always a real number.

A radical expression is in simplest form if no radicands contain fractions and no radicals appear in the denominator of a fraction. Similarly, a complex number is in simplest form if no imaginary numbers appear in the denominator of a fraction. You can use complex conjugates to simplify a fraction with a complex number in the denominator. This process is called **rationalizing the denominator**.

Example 4 Equate Complex Numbers

Find the values of x and y that make $5x - 7 + (y + 4)i = 13 + 11i$ true.

Use equations relating the real and imaginary parts to solve for x and y.

$5x - 7 = \underline{13}$ Real parts

$5x = \underline{20}$ Add 7 to each side.

$x = \underline{4}$ Divide each side by 5.

$y + 4 = \underline{11}$ Imaginary parts

$y = \underline{7}$ Subtract 4 from each side.

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Study Tip

These abbreviations represent the sets of real numbers.

Letter	Set
Q	rationals
I	irrationals
Z	integers
W	wholes
N	naturals

Think About It!

Compare and contrast the subsets of the complex number system using the Venn diagram.

Sample answer: For $a + bi$, if $b = 0$ then the number is real and can be irrational, rational, an integer, whole, and/or a natural number. If $b \neq 0$ then the number is imaginary. It is pure imaginary if $a = 0$.

Go Online You can watch a video to see how to add or subtract complex numbers.

Example 5 Add or Subtract Complex Numbers

Simplify $(8 + 3i) - (4 - 10i)$.

$(8 + 3i) - (4 - 10i) = (8 - \underline{4}) + [3 - (\underline{-10})]i$	Commutative and Associative Properties
$= \underline{4} + 13i$	Simplify.

Check

Simplify $(-5 + 5i) - (-3 + 8i)$.

$\underline{-2} + \underline{-3}\,i$

Study Tip

Imaginary Unit Complex numbers are often used with electricity. In these problems, j is usually used in place of i.

Example 6 Multiply Complex Numbers

ELECTRICITY **The voltage V of an AC circuit can be found using the formula $V = CI$, where C is current and I is impedance. If $C = 3 + 2j$ amps and $I = 7 - 5j$ ohms, determine the voltage.**

$V = CI$	Voltage Formula
$= (3 + 2j)(7 - 5j)$	$C = 3 + 2j$ and $I = 7 - 5j$
$= 3(\underline{7}) + 3(\underline{-5j}) + 2j(\underline{7}) + 2j(\underline{-5j})$	FOIL Method
$= \underline{21} - 15j + \underline{14j} - 10j^2$	Multiply.
$= 21 - \underline{j} - 10(\underline{-1})$	$j^2 = -1$
$= \underline{31} - j$	Add.

The voltage is $\underline{31 - j}$ volts.

Example 7 Divide Complex Numbers

Simplify $\frac{5i}{3 + 2i}$.

Rationalize the denominator to simplify the fraction.

$\frac{5i}{3 + 2i} = \frac{5i}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i}$	$3 + 2i$ and $3 - 2i$ are complex conjugates.
$= \frac{15i - 10i^2}{9 - 4i^2}$	Multiply the numerator and denominator.
$= \frac{15i - 10(-1)}{9 - 4(-1)}$	$i^2 = -1$
$= \frac{15i + 10}{13}$	Simplify.
$= \frac{10}{13} + \frac{15}{13}i$	$a + bi$ form

Check

Simplify $\frac{2i}{-4 + 3i}$. $\underline{\frac{6 - 8i}{25} \text{ or } \frac{6}{25} - \frac{8}{25i}}$

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