Solving Absolute Value Equations and Inequalities

Learn Solving Absolute Value Equations Algebraically

The **absolute value** of a number is its distance from zero on the number line. The definition of absolute value can be used to solve equations that contain absolute value expressions by constructing two cases. For any real numbers a and b, if $|a| = b$ and $b \geq 0$, then $a = b$ or $a = -b$.

Step 1 Isolate the absolute value expression on one side of the equation.

Step 2 Write the two cases.

Step 3 Use the properties of equality to solve each case.

Step 4 Check your solutions.

Absolute value equations may have one, two, or no solutions.

- An absolute value equation has one solution if one of the answers does not meet the constraints of the problem. Such an answer is called an **extraneous solution**.
- An absolute value equation has no solution if there is no answer that meets the constraints of the problem. The solution set of this type of equation is called the **empty set**, symbolized by {} or Ø.

Today's Goals

- Write and solve absolute value equations, and graph the solutions on a number line.
- Write and solve absolute value inequalities, and graph the solutions on a number line.

Today's Vocabulary

absolute value

extraneous solution

empty set

Example 1 Solve an Absolute Value Equation

Solve $2|5x + 1| - 9 = 4x + 17$. Check your solutions. Then graph the solution set.

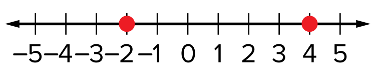
$2\lvert 5x + 1\rvert - 9 = 4x + 17$	Original equation
$2\lvert 5x + 1\rvert = 4x + \underline{26}$	Add 9 to each side.
$\lvert 5x + 1\rvert = \underline{2x} + \underline{13}$	Divide each side by 2.

Case 1	Case 2
$5x + 1 = \underline{2x + 13}$	$5x + 1 = \underline{-(2x + 13)}$
$3x + 1 = \underline{13}$	$7x + 1 = \underline{-13}$
$x = \underline{4}$	$x = \underline{-2}$

CHECK Substitute each value in the original equation.

$2\lvert 5(4) + 1\rvert - 9 \stackrel{?}{=} 4(4) + 17$	$2\lvert 5(-2) + 1\rvert - 9 \stackrel{?}{=} 4(-2) + 17$
$33 = 33$ True	$9 = 9$ True

Both solutions make the equation true. Thus, the solution set is {4, −2}. The solution set can be graphed by graphing each solution on a number line.



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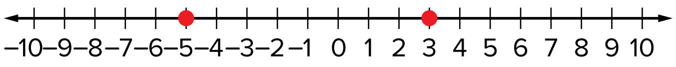
Watch Out!

Distribute the Negative For Case 2, remember to use the Distributive Property to multiply the entire expression on the right side of the equation by −1.

Your Notes

Check

Graph the solution set of $|5x - 3| - 6 = -2x + 12$.



Think About It!

What would the graph of the solution set of an absolute value equation with only one solution look like?

a single point graphed on a number line

Example 2 Extraneous Solution

Solve $2|x + 1| - x = 3x - 4$. Check your solutions.

$2|x + 1| - x = 3x - 4$ Original equation

$2|x + 1| = 4x - 4$ Add x to each side.

$|x + 1| = 2x - 2$ Divide each side by 2.

Case 1

$x + 1 = 2x - 2$

$1 = x - 2$

$3 = x$

Case 2

$x + 1 = -(2x - 2)$

$x + 1 = -2x + 2$

$3x + 1 = 2$

$x = \frac{1}{3}$

There appear to be two solutions, 3 and $\frac{1}{3}$.

CHECK Substitute each value in the original equation.

$2|3 + 1| - 3 \stackrel{?}{=} 3(3) - 4$

$5 = 5$ True

$2\left|\frac{1}{3} + 1\right| - \left(\frac{1}{3}\right) \stackrel{?}{=} 3\left(\frac{1}{3}\right) - 4$

$\frac{7}{3} \neq -3$ False

Because $\frac{7}{3} \neq -3$, the only solution is 3. Thus, the solution set is $\{3\}$.

Talk About It!

Is the following statement *always*, *sometimes*, or *never* true? Justify your argument. For real numbers a, b, and c, $|ax + b| = -c$ has no solution.

Sometimes; sample answer: If $c > 0$, then the equation has no solution, but if $c \leq 0$, the equation may have a solution.

Example 3 The Empty Set

Solve $|4x - 7| + 10 = 2$.

$|4x - 7| + 10 = 2$ Original equation

$|4x - 7| = -8$ Subtract 10 from each side.

Because the absolute value of a number is always positive or zero, this sentence is never true. The solution is $\varnothing$.

Check

Solve each absolute value equation.

a. $|x + 10| = 4x - 8$ $\{6\}$

b. $3|4x - 11| + 1 = 9x + 13$ $\{15, 1\}$

c. $|2x + 5| - 18 = -3$ $\{5, -10\}$

d. $-5|7x - 2| + 3x = 3x + 10$ $\varnothing$

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Example 4 Write and Solve an Absolute Value Equation

FOOTBALL **The NFL regulates the inflation, or air pressure, of footballs used during games. It requires that footballs have an air pressure of 13 pounds per square inch (PSI), plus or minus 0.5 PSI. What is the greatest and least acceptable air pressure of a regulation NFL football?**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the greatest and least acceptable air pressure for an NFL football. How can I write an absolute value equation to find the solution? Will there be one solution or two that make sense in this problem? I can find the answers to my questions by referencing other examples in the lesson and by checking my solutions.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will write an equation to represent the situation. I have learned how to write and solve an equation involving absolute value

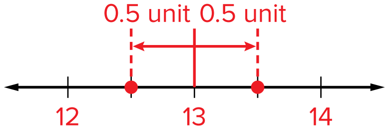
3 What is your solution?

Use your strategy to solve the problem.

What absolute value equation represents the greatest and least acceptable air pressure?

$|x - 13| = 0.5$

How are the solutions of the equation represented on a graph?



Interpret your solution. What are the greatest and least acceptable air pressures for an NFL football?

The greatest air pressure an NFL football can have is 13.5 PSI and the least is 12.5 PSI.

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution. Sample answer: Because the distance between 13 and each solution is 0.5, both solutions satisfy the constraints of the equation. I can substitute each solution back into the original equation and sure that the value makes the equation true. The pressures of 12.5 PSI and 13.5 PSI are within 0.5 PSI of 13 PSI.

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Learn Solving Absolute Value Inequalities Algebraically

When solving absolute value inequalities, there are two cases to consider. These two cases can be rewritten as a compound inequality.

Key Concept • Absolute Value Inequalities

For all real numbers a, b, c and x, $c > 0$, the following statements are true.

Absolute Value Inequality	Case 1	Case 2	Compound Inequality
$\lvert ax + b \rvert < c$	$ax + b < c$	$-(ax + b) < c$ $\frac{-(ax+b)}{-1} > \frac{c}{-1}$ $ax + b > -c$	$ax + b < c$ and $ax + b > -c$ OR $-c < ax + b < c$
$\lvert ax + b \rvert > c$	$ax + b > c$	$-(ax + b) > c$ $\frac{-(ax+b)}{-1} < \frac{c}{-1}$ $ax + b < -c$	$ax + b > c$ or $ax + b < -c$

These statements are also true for $\leq$ and $\geq$, respectively.

Think About It!

Describe a shortcut you could use to write case 2.

Sample answer: Reverse the inequality symbol, and multiply the side of the inequality that does not contain the absolute value expression by −1.

Example 5 Solve an Absolute Value Inequality ($<$ or $\leq$)

Solve $|4x - 8| - 5 < 11$. Then graph the solution set.

$|4x - 8| - 5 < 11$ Original inequality

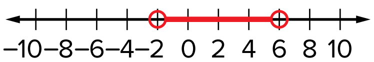
$|4x - 8| < \underline{16}$ Add 5 to each side.

Since the inequality uses $<$, rewrite it as a compound inequality joined by the word *and*. For the case where the expression inside the absolute value symbols is negative, reverse the inequality symbol.

$4x - 8 < 16$	and	$4x - 8 > -16$
$4x < \underline{24}$		$4x > \underline{-8}$
$x < \underline{6}$		$x > \underline{-2}$

So, $x \underline{<} 6$ and $x \underline{>} -2$. The solution set is $\{x \mid \underline{-2} < x < \underline{6}\}$. All values of x between -2 and 6 satisfy the original inequality.

The solution set represents the interval between two numbers. Since the $<$ symbols indicate that -2 and 6 are not solutions, graph the endpoints of the interval on a number line using circles. Then, shade the interval from -2 to 6.



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An alternate method is available for this example.

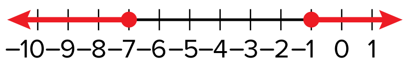
Study Tip

Check Your Solutions Remember to substitute your solutions back into the original inequality to check that they make that inequality true.

Check

Solve $6|x + 4| - 3 \geq 15$. Then graph the solution set.

$x \leq \underline{-7}$ or $x \geq \underline{-1}$



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Example 6 Solve an Absolute Value Inequality (> and ≥)

Solve the inequality $\frac{|6x+3|}{2} + 5 \geq 14$. Then graph the solution set.

$\frac{\lvert 6x+3 \rvert}{2} + 5 \geq 14$	Original inequality
$\frac{\lvert 6x+3 \rvert}{2} \geq 9$	Subtract 5 from each side.
$\lvert 6x+3 \rvert \geq 18$	Multiply each side by 2.

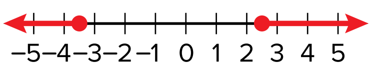
Since the inequality uses ≥, rewrite it as a compound inequality joined by the word *or*. For the case where the expression inside the absolute value symbols is negative, reverse the inequality symbol.

$6x + 3 \geq 18$	or	$6x + 3 \leq -18$
$6x \geq 15$		$6x \leq -21$
$x \geq \frac{5}{2}$		$x \leq -\frac{7}{2}$

So, $x \geq \frac{5}{2}$ and $x \leq -\frac{7}{2}$. The solution set is $\left\{x \,\middle|\, x \leq -\frac{7}{2} \text{ or } x \geq \frac{5}{2}\right\}$.

All values of x less than or equal to $-\frac{7}{2}$ as well as values of x greater than $\frac{5}{2}$ satisfy the constraints of the original inequality.

The solution set represents the union of two intervals. Since the ≤ and ≥ symbols indicate that $-\frac{7}{2}$ and $\frac{5}{2}$ are solutions, graph the endpoints of the interval on a number line using dots. Then, shade all points less than $-\frac{7}{2}$ and all points greater than $\frac{5}{2}$.



Watch Out!

Isolate the Expression Remember to isolate the absolute value expression on one side of the inequality symbol before determining whether to rewrite an absolute value inequality using *and* or *or*. When transforming the inequality, you might divide or multiply by a negative number, causing the inequality symbol to be reversed.

Check

Match each solution set with the appropriate absolute value inequality.

$-8|x + 14| + 7 \geq -17$ F

$\frac{|2x - 8|}{3} - 10 < 6$ E

$5|2x + 28| + 6 \leq -24$ A

$\frac{|3x - 12|}{4} - 13 > 5$ C

A. $\{x \mid x \leq -17 \text{ or } x \geq -11\}$ **B.** $\{x \mid 28 < x < -20\}$

C. $\{x \mid x < -20 \text{ or } x > 28\}$ **D.** $\{x \mid x \leq -11 \text{ or } x \geq -17\}$

E. $\{x \mid -20 < x < 28\}$ **F.** $\{x \mid -17 \leq x \leq -11\}$

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Example 7 Write and Solve an Absolute Value Inequality

SLEEP **You can find how much sleep you need by going to sleep without turning on an alarm. Once your sleep pattern has stabilized, record the amount of time you spend sleeping each night. The amount of time you sleep plus or minus 15 minutes is your sleep need. Suppose you sleep 8.5 hours per night. Write and solve an inequality to represent your sleep need, and graph the solution on a number line.**

Part A Write an absolute value inequality to represent the situation.

The difference between your actual sleep need and the amount of time you sleep is less than or equal to 15 minutes. So, 8.5 hours is the central value and 15 minutes, or 0.25 hour, is the acceptable range.

The difference between your actual sleep need and 8.5 hours is 0.25 hour. Let n = your actual sleep need.

$$|n - 8.5| \leq 0.25$$

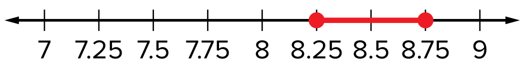
Part B Solve the inequality and graph the solution set.

Rewrite $|n - 8.5| \leq 0.25$ as a compound inequality.

$n - 8.5 \leq 0.25$ and $n - 8.5 \geq -0.25$

$n \leq 8.75$ $\quad$ $n \geq 8.25$

The solution set represents the interval between two numbers. Since the $\leq$ and $\geq$ symbols indicate that 8.25 and 8.75 are solutions, graph the endpoints of the interval on a number line using dots. Then, shade the interval from 8.25 to 8.75.



This means that you need between 8.25 and 8.75 hours of sleep per night, inclusive.

Check

FOOD A survey found that 58% of American adults eat at a restaurant at least once a week. The margin of error was within 3 percentage points.

Part A Write an absolute value inequality to represent the range of the percent of American adults who eat at a restaurant once a week, where x is the actual percent. $|x - 58| \leq 3$

Part B Use your inequality from Part **A** to find the range of the percent of American adults who eat at a restaurant once a week.

The actual percent of American adults who eat out at least once a week is $\{x \mid 55 \leq x \leq 61\}$.

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