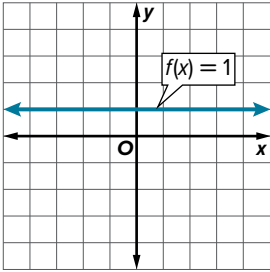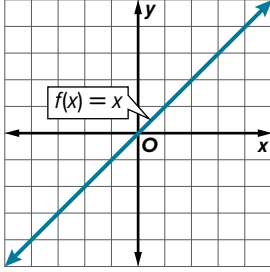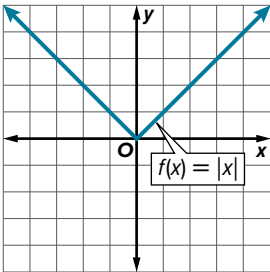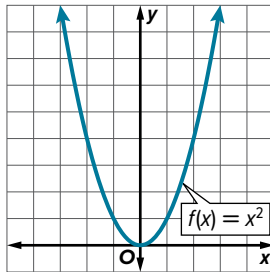# Transformations of Functions

## Learn Translations of Functions

A **family of graphs** includes graphs and equations of graphs that have at least one characteristic in common. The parent graph is transformed to create other members in a family of graphs.

### Key Concept • Parent Functions

| Constant Function | Identity Function |
|---|---|
|  |  |
| The general equation of a **constant function** is $f(x) = a$, where $a$ is any number.<br>Domain: all real numbers<br>Range: $\{f(x) \mid f(x) = a\}$ | The **identity function** $f(x) = x$ includes all points with coordinates $(a, a)$. It is the parent function of most linear functions.<br>Domain: all real numbers<br>Range: all real numbers |
| **Absolute Value Function** | **Quadratic Function** |
|  |  |
| The parent function of absolute value functions is $f(x) = \|x\|$.<br>Domain: all real numbers<br>Range: $\{f(x) \mid f(x) \geq 0\}$ | The parent function of quadratic functions is $f(x) = x^2$.<br>Domain: all real numbers<br>Range: $\{f(x) \mid f(x) \geq 0\}$ |

A **translation** is a transformation in which a figure is slid from one position to another without being turned.

### Key Concept • Translations

| Translation | Change to Parent Graph |
|---|---|
| $f(x) + k$; $k > 0$ | The graph is translated $k$ units up. |
| $f(x) + k$; $k < 0$ | The graph is translated $\|k\|$ units down. |
| $f(x - h)$; $h > 0$ | The graph is translated $h$ units right. |
| $f(x - h)$; $h < 0$ | The graph is translated $\|h\|$ units left. |

### Today's Goals
- Apply translations to the graphs of functions.
- Apply dilations to the graphs of functions.
- Apply compositions of transformations to the graphs of functions and use transformations to write equations from graphs.

### Today's Vocabulary
family of graphs
constant function
identity function
transformations
translation
dilation
reflection
line of reflection

**Go Online**
You may want to complete the Concept Check to check your understanding.

**Go Online**
You can watch a video to see how to describe translations of functions.

**Think About It!**
Describe the vertex and axis of symmetry of a translated quadratic function in terms of $h$ and $k$.

Sample answer: The vertex of a translated quadratic function is at ($h$, $k$) and the line of symmetry is $x = h$.

## Example 1 Translations

**Describe the translation in $g(x) = (x + 2)^2 - 4$ as it relates to the graph of the parent function.**

Since $g(x)$ is quadratic, the parent function is $f(x) = \underline{x^2}$.

Since $f(x) = x^2$, $g(x) = f(x - h) + k$, where $h = \underline{-2}$ and $k = \underline{-4}$.

The constant $k$ is added to the function after it has been evaluated, so $k$ affects the output, or $y$-values. The value of $k$ is less than 0, so the graph of $f(x) = x^2$ is translated $\underline{\text{down}}$ 4 units.

The value of $h$ is subtracted from $x$ before it is evaluated and is less than 0, so the graph of $f(x) = x^2$ is also translated 2 units $\underline{\text{left}}$.

The graph of $g(x) = (x + 2)^2 - 4$ is the translation of the graph of the parent function $\underline{2}$ units left and 4 units $\underline{\text{down}}$.
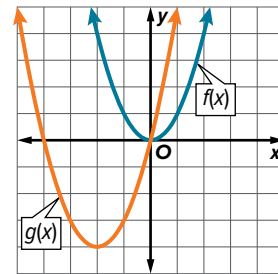
## Example 2 Identify Translated Functions from Graphs

**Use the graph of the function to write its equation.**

The graph is an absolute value function with a parent function of $\underline{f(x) = |x|}$. Notice that the vertex of the function has been shifted both vertically and horizontally from the parent function.

To write the equation of the graph, determine the values of $h$ and $k$ in $g(x) = |x - h| + k$.

The translated graph has been shifted 2 units up and 1 unit right. So, $h = \underline{1}$ and $k = \underline{2}$. Thus, $g(x) = \underline{|x - 1| + 2}$.
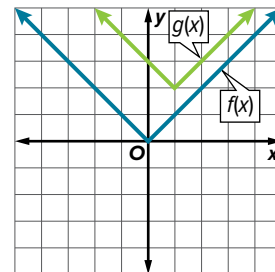
## Learn Dilations and Reflections of Functions

A **dilation** is a transformation that stretches or compresses the graph of a function. Multiplying a function by a constant dilates the graph with respect to the $x$- or $y$-axis.

**Key Concept • Dilations**

| Dilation | Change to Parent Graph |
|---|---|
| $af(x)$, $|a| > 1$ | The graph is stretched vertically. |
| $af(x)$, $0 < |a| < 1$ | The graph is compressed vertically. |
| $f(ax)$, $|a| > 1$ | The graph is compressed horizontally. |
| $f(ax)$, $0 < |a| < 1$ | The graph is stretched horizontally. |

**Go Online** You can complete an Extra Example online.

A **reflection** is a transformation where a figure, line, or curve, is flipped in a **line of reflection**. Often the reflection is in the $x$- or $y$-axis.

When a parent function $f(x)$ is multiplied by $-1$, the result $-f(x)$ is a reflection of the graph in the $x$-axis. When only the variable is multiplied by $-1$, the result $f(-x)$ is a reflection of the graph in the $y$-axis.

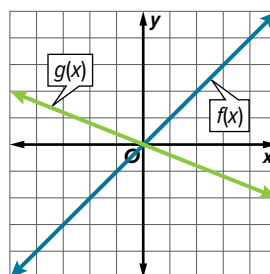| Key Concept • Reflections | |
|---|---|
| **Reflection** | **Change to Parent Graph** |
| $-f(x)$ | reflection in the $x$-axis |
| $f(-x)$ | reflection in the $y$-axis |

## Example 3  Vertical Dilations

**Describe the dilation and reflection in**

$g(x) = -\frac{2}{5}x$ **as it relates to the parent function.**

Since $g(x)$ is a linear function, the parent

function is $f(x) = \underline{\ x\ }$.

Since $f(x) = x$, $g(x) = -1 \cdot a \cdot f(x)$ where

$a = \underline{\ \frac{2}{5}\ }$.

The function is multiplied by $-1$ and the constant $a$ after it has been evaluated. $0 < |a| < 1$, so the graph is compressed vertically and reflected in the $x$-axis.

The graph of $g(x) = -\frac{2}{5}x$ is the graph of the parent function

__compressed__ vertically and reflected in the __x-axis__.

## Example 4  Horizontal Dilations

**Describe the dilation and reflection in $g(x) = (-2.5x)^2$ as it relates to the parent function.**

Since $g(x)$ is a quadratic function, the parent

function is $f(x) = \underline{\ x^2\ }$. Since $f(x) = x^2$,

$g(x) = f(-1 \cdot a \cdot x)$, where $a = \underline{\ 2.5\ }$.

$x$ is multiplied by $\underline{\ -1\ }$ and the constant $a$ before the function is performed and $|a|$ is greater than 1, so the graph of $f(x) = x^2$ is compressed __horizontally__ and reflected in the __y-axis__.
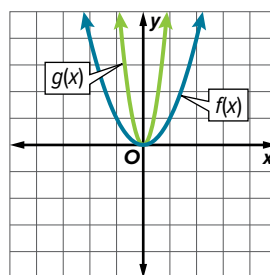
▶ **Go Online** You can complete an Extra Example online.

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## Explore Using Technology to Transform Functions

**Online Activity** Use graphing technology to complete the Explore.

> ⊗
>
> **Ⓠ INQUIRY** How does performing an operation on a function change its graph?
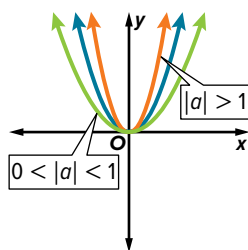
## Learn Transformations of Functions

The general form of a function is $g(x) = a \cdot f(x - h) + k$, where $f(x)$ is the parent function. Each constant in the equation affects the parent graph.

- The value of $|a|$ stretches or compresses (dilates) the parent graph.

- When the value of $a$ is negative, the graph is reflected across the $x$-axis.

- The value of $h$ shifts (translates) the parent graph left or right.

- The value of $k$ shifts (translates) the parent graph up or down.

### Key Concept • Transformations of Functions

| | |
|---|---|
| **Dilation, $a$** | **Reflection, $a$** |
| If $|a| > 1$, the graph of $f(x)$ is stretched vertically. If $0 < |a| < 1$, the graph of $f(x)$ is compressed vertically. | If $a > 0$, the graph of $f(x)$ opens up. If $a < 0$, the graph of $f(x)$ opens down. |





| | |
|---|---|
| **Horizontal Translation, $h$** | **Vertical Translation, $k$** |
| If $h > 0$, the graph of $f(x)$ is translated $h$ units right. If $h < 0$, the graph of $f(x)$ is translated $|h|$ units left. | If $k > 0$, the graph of $f(x)$ is translated $k$ units up. If $k < 0$, the graph of $f(x)$ is translated $|k|$ units down. |
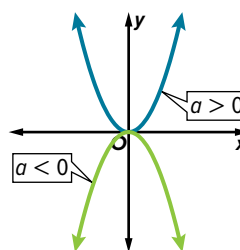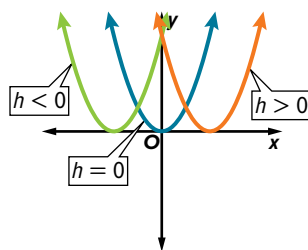
## Example 5  Multiple Transformations of Functions

**Describe how the graph of $g(x) = -\frac{2}{3}|x + 3| + 1$ is related to the graph of the parent function.**
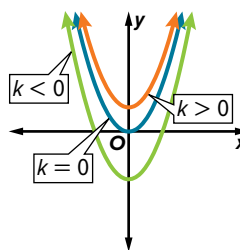
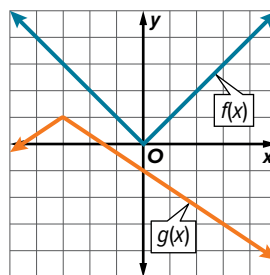The parent function is $f(x) = |x|$.

Since $f(x) = |x|$, $g(x) = af(x - h) + k$ where

$a = -\frac{2}{3}$, $h = -3$ and $k = 1$.

The graph of $g(x) = -\frac{2}{3}|x + 3| + 1$ is the graph

of the parent function __compressed__

vertically, reflected in the __x-axis__, and

translated 3 units __left__ and 1 unit __up__.

### Check

Describe how $g(x) = -(0.4x + 2)$ is related to the graph of the parent function. __graph of $f(x) = x$ stretched horizontally, reflected in the x-axis, and translated 2 units left__

## Example 6  Apply Transformations of Functions

**DOLPHINS**  **Suppose the path of a dolphin during a jump is modeled by $g(x) = -0.125(x - 12)^2 + 18$, where $x$ is the horizontal distance traveled by the dolphin and $g(x)$ is its height above the surface of the water. Describe how $g(x)$ is related to its parent function and interpret the function in the context of the situation.**

Because $f(x) = x^2$ is the parent function, $g(x) = af(x - h) + k$, where

$a = $ __−0.125__, $h = $ __12__, and $k = $ __18__.

**Translations**

$12 > 0$, so the graph of $f(x) = x^2$ is translated 12 units __right__.

$18 > 0$, so the graph of $f(x) = x^2$ is translated 18 units __up__.

**Dilation and Reflection**

$0 < |-0.125| < 1$, so the graph of $f(x) = x^2$ is __compressed__ vertically.

$a < 0$, so the graph of $f(x) = x^2$ is a __reflection__ in the $x$-axis.

**Interpret the Function**

Because $a$ is negative, the path of the dolphin is modeled by a parabola that opens __down__. This means that the vertex of the parabola $(h, k)$ represents the __maximum height__ of the dolphin, 18 feet, at 12 feet from the starting point of the jump.

**Go Online**  You can complete an Extra Example online.

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**Go Online**
You can watch a video to see how to use transformations to graph an absolute value function.

**Think About It!**
Write an equation for a quadratic function that opens down, has been stretched vertically by a factor of 4, and is translated 2 units right and 5 units down.

Sample answer:
$g(x) = -4(x - 2)^2 - 5$

**Study Tip**
**Interpretations**  When interpreting transformations, analyze how each value influences the function and alters the graph. Then determine what you think each value might mean in the context of the situation.
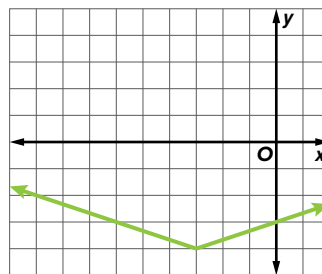
## Example 7  Identify an Equation from a Graph

**Write an equation for the function.**

**Step 1  Analyze the graph.**

The graph is an absolute value function
with a parent function of $f(x) = |x|$.
Analyze the graph to make a prediction
about the values of $a$, $h$, and $k$ in the
equation $y = a|x - h| + k$.

The graph appears to be wider than the parent function, implying a
vertical __compression__, and is not reflected. So, $a$ is __positive__ and
$0 < |a| < 1$.

The graph has also been shifted left and down from the parent graph.
So, $h$ __<__ 0 and $k$ __<__ 0.

**Step 2  Identify the translation(s).**

Identify the horizontal and vertical translations to find the values of $h$ and $k$.

The vertex is shifted 3 units left, so $h = $ __−3__.

It is also shifted 4 units down, so $k = $ __−4__.

| | |
|---|---|
| $y = a\|x - h\| + k$ | General form of the equation |
| $y = a\|x - \underline{(-3)}\| + \underline{(-4)}$ | $h = -3$ and $k = -4$ |
| $y = a\|x \underline{+ 3}\| \underline{- 4}$ | Simplify. |

**Step 3  Identify the dilation and/or reflection.**

Use the equation from Step 2 and a point on the graph to find the
value of $a$.

The point $(0, -3)$ lies on the graph. Substitute the coordinates in for $x$
and $y$ to solve for $a$.

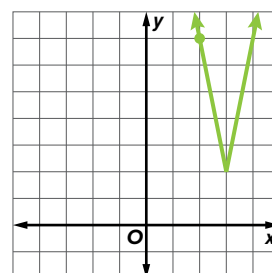| | |
|---|---|
| $y = a\|x + 3\| - 4$ | Original equation |
| $\underline{-3} = a\|\underline{0} + 3\| - 4$ | $(0, -3) = (x, y)$ |
| $-3 = a\|\underline{3}\| - 4$ | Add. |
| $-3 = \underline{3}a - 4$ | Evaluate the absolute value. |
| $\underline{\frac{1}{3}} = a$ | Solve. |

**Step 4  Write an equation for the function.**

Since $a = \frac{1}{3}$, $h = -3$ and $k = -4$, the
equation is $g(x) = \underline{\frac{1}{3}|x + 3| - 4}$.

## Check

Use the graph to write an equation for $g(x)$.

$\underline{g(x) = 5|x - 3| + 2}$

**Go Online**  You can complete an Extra Example online.

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**Watch Out!**

**Choosing a Point**
When substituting for $x$
and $y$ in the equation,
use a point other than
the vertex.

**Think About It!**

How does the equation
you found compare to
the prediction you
made in step 1?

Sample answer: The
values of $a$, $h$, and $k$
match my prediction:
$a = \frac{1}{3}$ and $0 < \frac{1}{3} < 1$;
$h = -3$ and $-3 < 0$;
$k = -4$ and $-4 < 0$.