Functions and Continuity

Explore Analyzing Functions Graphically

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you use a graph to analyze the relationship between the domain and range of a function?

Explore Defining and Analyzing Variables

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How can you define variables to effectively model a situation?

Learn Functions

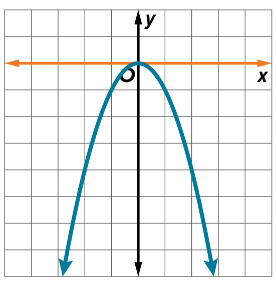
A function describes a relationship between input and output values. The **domain** is the set of *x*-values to be evaluated by a function. The **codomain** is the set of all the *y*-values that could possibly result from the evaluation of the function. The codomain of a function is assumed to be all real numbers unless otherwise stated. The **range** is the set of *y*-values that actually result from the evaluation of the function. The range is contained within the codomain.

If each element of a function's range is paired with exactly one element of the domain, then the function is a **one-to-one function**. If a function's codomain is the same as its range, then the function is an **onto function**.

Example 1 Domains, Codomains, and Ranges

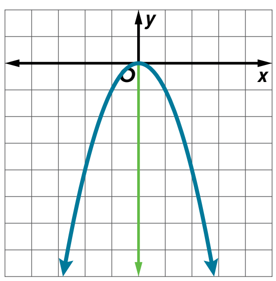
Part A Identify the domain, range, and codomain of the graph.

Domain



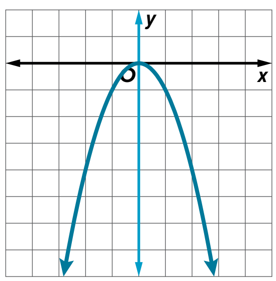
Because there are no restrictions on the *x*-values, the domain is all real numbers.

Range



Because the maximum *y*-value is 0, the range is $y \leq$ 0.

Codomain



Because it is not stated otherwise, the codomain is all real numbers.

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Today's Goals

- Determine whether functions are one-to-one and/or onto.
- Determine the continuity, domain, and range of functions.
- Write the domain and range of functions by using set-builder and interval notations.

Today's Vocabulary

domain
codomain
range
one-to-one function
onto function
continuous function
discontinuous function
discrete function
algebraic notation
set-builder notation
interval notation

Study Tip

Horizontal Line Test Performing the horizontal line test can help you examine a function. Place a pencil at the top of the graph and slowly move it down the graph to represent a horizontal line. If there are places where pencil intersects the graph at more than one point at a time, then it is not one-to-one. If there are places where the pencil does not intersect the graph at all, then it is not onto. Consider these two results to determine if the function is one-to-one, onto, both, or neither.

Your Notes

Part B Use these values to determine whether the function is onto.

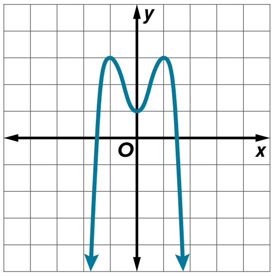
The range is not the same as the codomain because it does not include the positive real numbers. Therefore, the function is not onto.

Check

For what codomain is $f(x)$ an onto function? A

A. $y \leq 3$ **B.** $y \geq 3$

C. all real numbers **D.** $x \leq 3$



Example 2 Identify One-to-One and Onto Functions from Tables

OLYMPICS **The table shows the number of medals the United States won at five Summer Olympic Games.**

Year	Number of Gold Medals	Number of Silver Medals	Number of Bronze Medals
2016	46	37	38
2012	46	29	29
2008	36	38	36
2004	36	39	26
2000	37	24	32

Analyze the functions that give the number of gold and silver medals won in a particular year. Define the domain and range of each function and state whether it is *one-to-one, onto, both* or *neither*.

Gold Medals	Silver Medals
Let $f(x)$ be the function that gives the number of gold medals won in a particular year. The domain is in the column Year, and the range is in the column Number of Gold Medals. The function is not one-to-one because two values in the domain, 2016 and 2012, share the same value in the range, 46, and two values in the domain, 2008 and 2004, share the same value in the range, 36. The function is not onto because the range does not include every whole number.	Let $g(x)$ be the function that gives the number of silver medals won in a particular year. The domain is the column Year, and the range is the column Number of Silver Medals. The function is one-to-one because no two values in the domain share a value in the range. The function is not onto because the range does not include every whole number.

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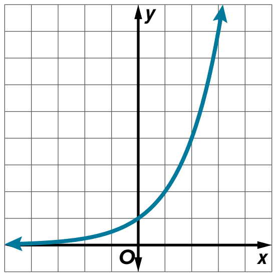
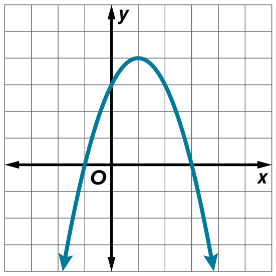
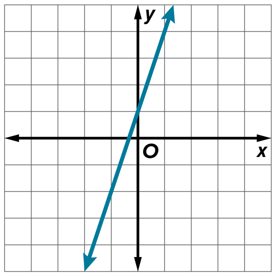
Use a Source

Choose another country and research the number of medals they won in the Summer Olympic Games from 2000-2016. Are the functions that give the number of each type of medal won in a particular year *one-to-one, onto, both,* or *neither*?

Sample answer: The functions that represent the number of each type of medal won by Great Britain given the year are all one-to-one but not onto.

Example 3 Identify One-to-One and Onto Functions from Graphs

Determine whether each function is *one-to-one, onto, both,* or *neither* for the given codomain.

***f*(*x*), where the codomain is all real numbers** 	The graph indicates that the domain is all real numbers, and the range is all positive real numbers. Every x-value is paired with exactly one unique y-value, so the function is one-to-one. If the codomain is all real numbers, then the range is not equal to the codomain. So, the function is not onto.
***g*(*x*), where the codomain is $\{y \mid y \leq 4\}$** 	The graph indicates that the domain is all real numbers, and the range is $y \leq 4$. Each x-value is not paired with a unique y-value; for example, both $x = 0$ and $x = 2$ are paired with $y = 3$. So the function is not one-to one. The codomain and range are equal, so the function is onto.
***h*(*x*), where the codomain is all real numbers** 	The graph indicates that the domain and range are both all real numbers. Every x-value is paired with exactly one unique y-value, so the function is one-to-one. The codomain are equal, so the function is onto.

Learn Discrete and Continuous Functions

Functions can be discrete, continuous, or neither. Real-world situations where only some numbers are reasonable are modeled by discrete functions. Situations where all real numbers are reasonable are modeled by continuous functions.

A **continuous function** is graphed with a line or an unbroken curve. A function that is not continuous is a **discontinuous function**. A **discrete function** is a discontinuous function in which the points are not connected. A function that is neither discrete nor continuous may have a graph in which some points are connected, but it is not continuous everywhere.

Study Tip

Intervals An interval is the set of all real numbers between two given numbers. For example, the interval $-2 < x < 5$ includes all values of x greater than -2 but less than 5. Intervals can also continue on infinitely in a direction. For example, the interval $y \geq 1$ includes all values of y greater than or equal to 1. You can use intervals to describe the values of x or y for which a function exists.

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Talk About It

Does the range of the function need to be all real numbers for a function to be continuous? Justify your argument.

No; sample answer: as long as neither the domain nor the range have any discontinuities, the function is continuous.

Problem-Solving Tip

Use a Graph If you are having trouble determining the continuity given the equation of a function, you can graph the function to help visualize the situation.

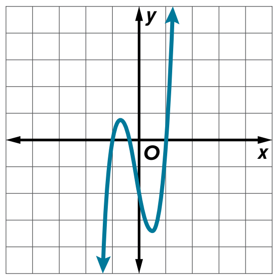
Study Tip

Accuracy When calculating cost, the result can be any fraction of a dollar or cent, and is therefore continuous. However, because the smallest unit of currency is \$0.01, the price you actually pay is rounded to the nearest cent. Therefore, the price you pay is discrete.

Example 4 Determine Continuity from Graphs

Examine the functions. Determine whether each function is *discrete*, *continuous*, or *neither* discrete nor continuous. Then state the domain and range of each function.

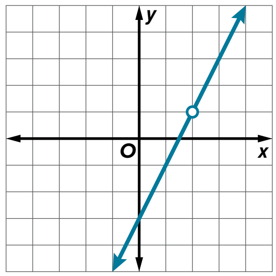
a. ***f*(*x*)**



The function is continuous because it is a curve with no breaks or discontinuities.

Because you can assume that the function continues forever, the domain and range are both all real numbers.

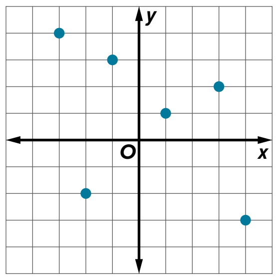
b. ***g*(*x*)**



The function is neither because there are continuous sections, but there is a break at (2, 1).

Because the function is not defined for $x = 2$, the domain is all values of x except $x = 2$. The function is not defined for $y = 1$, so the range is all values of y except $y = 1$.

c. ***h*(*x*)**



The function is discrete because it is made up of distinct points that are not connected.

The domain is $\{-3, -2, -1, 1, 3, 4\}$ and the range is $\{-3, -2, 1, 2, 3, 4\}$.

Example 5 Determine Continuity

BUSINESS **Determine whether the function that models the cost of coffee beans is *discrete, continuous*, or *neither* discrete nor continuous. Then state the domain and range of the function.**

Weight	Price
Up to 2 lbs	\$8/lb
2.5 lbs	\$20
3 lbs	\$22
5 lbs	\$35

Because customers can purchase any amount of coffee up to 2 pounds, the function is continuous over the interval $0 \le x \le$ 2.

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For larger quantities, the coffee is sold by distinct amounts. This part of the function is discrete.

Since the domain and range are made up of neither a single interval nor individual points, the function is neither discrete nor continuous.

The domain of the function is $0 \leq x \leq 2$ or $x = 2.5$, 3, 5. This represents the possible weights of coffee beans that customers could purchase. The range of the function is $0 \leq y \leq$ 16 or $y = 20$, 22, 35. This represents the possible costs of coffee beans.

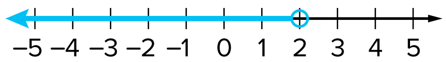
Think About It!

Why does the range include values from 0 to 16 instead of 0 to 8?

Sample answer: The price of coffee beans is \$8 per pound for up to 2 pounds, so customers pay up to $2 \cdot \$8 = \$16$.

Learn Set-Builder and Interval Notation

Sets of numbers like the domain and range of a function can be described by using various notations. Set-builder notation, interval notation, and algebraic notation are all concise ways of writing a set of values. Consider the set of values represented by the graph.



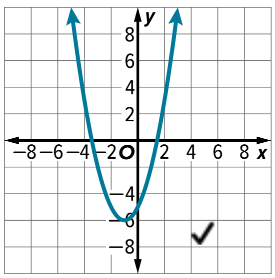
- In **algebraic notation**, sets are described using algebraic expressions. Example: $x < 2$
- **Set-builder notation** is similar to algebraic notation. Braces indicate the set. The symbol | is read as *such that*. The symbol ∈ is read *is an element of*. Example: $\{x | x < 2\}$
- In **interval notation** sets are described using endpoints with parentheses or brackets. A parenthesis, (or), indicates that an endpoint *is not* included in the interval. A bracket, [or], indicates that an endpoint is included in the interval. Example: $(-\infty, 2)$

Study Tip

Using Symbols You can use the symbol $\mathbb{R}$ to represent all real numbers in set-builder notation. In interval notation, the symbol ∪ indicates the union of two sets. Parentheses are always used with ∞ and $-\infty$ because they do not include endpoints.

Example 6 Set-Builder and Interval Notation for Continuous Intervals

Write the domain and range of the graph in set-builder and interval notation.



Domain

The graph will extend to include all x-values.

The domain is all real numbers.

$\{x \mid x \in \mathbb{R}\}$

$(-\infty, \infty)$

Range

The least y-value for this function is -6.

The range is all real numbers greater than or equal to -6.

$\{y \mid y \geq -6\}$

$[-6, \infty)$

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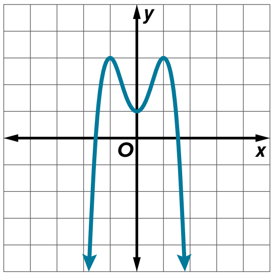
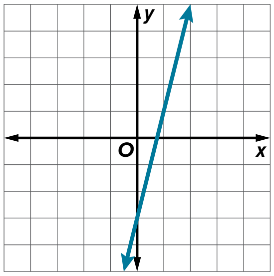
Check

State the domain and range of each graph in set-builder and interval notation.



D: $\{x \mid x \in \mathbb{R}\}$ R: $\{y \mid y \leq 3\}$

D: $(-\infty, \infty)$ R: $(-\infty, 3)$

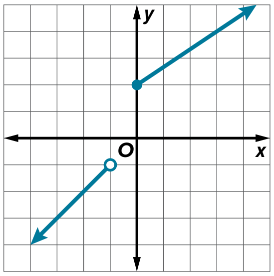


D: $\{x \mid x \in \mathbb{R}\}$ R: $\{y \mid y \in \mathbb{R}\}$

D: $\{-\infty, \infty\}$ R: $\{-\infty, \infty\}$

Example 7 Set-Builder and Interval Notation for Discontinuous Intervals

Write the domain and range of the graph in set-builder and interval notation.



Domain

The domain is all real numbers __less than__ -1 or __greater than or equal to__ 0.

$\{x \mid x$ __<__ -1 or x __$\geq$__ $0\}$

$(-\infty, -1$__)__ $\cup [0, \infty$__)__

Range

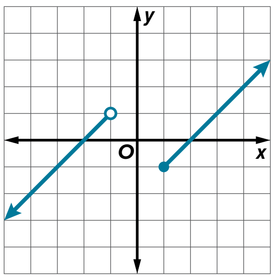
The range is all real numbers __less than__ -1 or __greater than or equal to__ 2.

$\{y \mid y$ __<__ -1 or y __$\geq$__ 2__}__

$(-\infty, -1$__)__ $\cup [2, \infty)$

Check

State the domain and range of the graph in set-builder and interval notation.



D: $\{x \mid x < -1 \text{ or } x \geq 1\}$ R: $\{y \mid y \in \mathbb{R}\}$

D: $(-\infty, -1) \cup [1, \infty)$ R: $(-\infty, \infty)$

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