Inverse Relations and Functions

Explore Inverse Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY For what values of n will $f(x) = x^n$ have an inverse that is also a function?

Learn Inverse Relations and Functions

Two relations are **inverse relations** if one relation contains elements of the form (a, b) when the other relation contains the elements of the form (b, a).

Two functions f and g are **inverse functions** if and only if both of their compositions are the identity function.

Key Concepts • Inverse Functions

Words: If f and f^{-1} are inverses, then $f(a) = b$ if and only if $f^{-1}(b) = a$.

Example: Let $f(x) = x - 5$ and represent its inverse as $f^{-1}(x) = x + 5$.

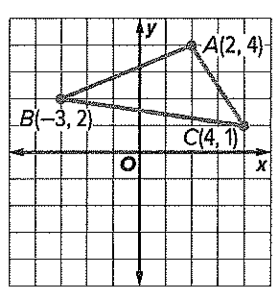
Evaluate $f(7)$.	Evaluate $f^{-1}(2)$.
$f(x) = x - 5$	$f^{-1}(x) = x + 5$
$f(7) = 7 - 5$ or 2	$f^{-1}(2) = 2 + 5$ or 7

Not all functions have an inverse function. If a function fails the horizontal line test, you can restrict the domain of the function to make the inverse a function. Choose a portion of the domain on which the function is one-to-one. There may be more than one possible domain.

Example 1 Find an Inverse Relation

GEOMETRY **The vertices of $\triangle ABC$ can be represented by the relation $\{(2, 4), (-3, 2), (4, 1)\}$. Find the inverse of the relation. Graph both the original relation and its inverse.**

Step 1 Graph the relation.
Graph the ordered pairs and connect the points to form a triangle.



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Today's Goals
- Find inverses of relations.
- Verify that two relations are inverses by using compositions.

Today's Vocabulary
inverse relations
inverse functions

Think About It!
Write a function that does not pass the horizontal line test.

Go Online You can complete an Extra Example online.

Your Notes

Think About It!
Describe the graph of the inverse relation.

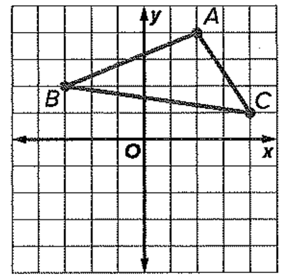
Study Tip
Inverses If $f^{-1}(x)$ is the inverse of $f(x)$, the graph of $f^{-1}(x)$ will be a reflection of the graph of $f(x)$ in the line $y = x$.

Go Online
You can learn how to graph a relation and its inverse on a graphing calculator by watching the video online.

Step 2 Find the inverse.

To find the inverse, exchange the coordinates of the ordered pairs.
{(4, ____), (2, ____), (1, ____)}

Step 3 Graph the inverse.



Example 2 Inverse Functions

Find the inverse of $f(x) = 3x + 2$. Then graph the function and its inverse.

Step 1 Rewrite the function.

Rewrite the function as an equation relating x and y.

$f(x) = 3x + 2 \rightarrow y = 3x + 2$

Step 2 Exchange x and y.

Exchange x and y in the equation.

$\_\_\_ = 3\_\_\_ + 2$

Step 3 Solve for y.

$$x = 3y + 2$$
$$x - 2 = 3y$$
$$\_\_\_\_ = y$$

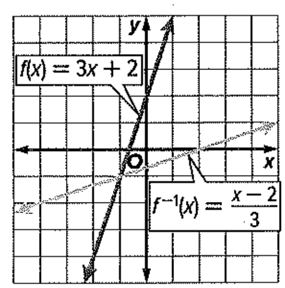
Step 4 Replace y with $f^{-1}(x)$.

Replace y with $f^{-1}(x)$ in the equation.

$y = \frac{x-2}{3} \rightarrow \_\_\_\_\_ = \frac{x-2}{3}$

The inverse of $f(x) = 3x + 2$ is $f^{-1}(x) = \frac{x-2}{3}$.

Step 5 Graph $f(x)$ and $f^{-1}(x)$.



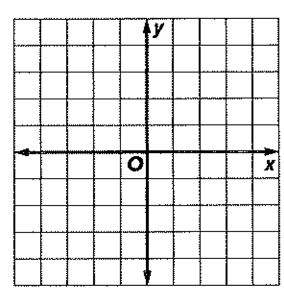
Check

Examine $f(x) = -\frac{1}{2}x + 1$.

Part A Find the inverse of $f(x) = -\frac{1}{2}x + 1$.

$f^{-1}(x) = \_\_\_\_\_\_\_$

Part B Graph $f(x) = -\frac{1}{2}x + 1$ and its inverse.



Go Online You can complete an Extra Example online.

Example 3 Inverses with Restricted Domains

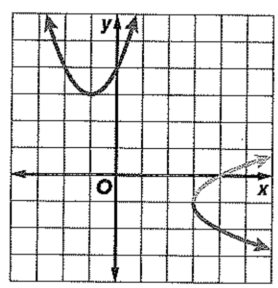
Examine $f(x) = x^2 + 2x + 4$.

Part A Find the inverse of $f(x)$.

$f(x) = x^2 + 2x + 4$	Original function
$\_\_\_ = x^2 + 2x + 4$	Replace $f^{-1}(x)$ with y.
$\_\_\_ = \_\_\_^2 + 2\_\_\_ + 4$	Exchange x and y.
$x - 4 = y^2 + 2y$	Subtract 4 from each side.
$x - 4 + 1 = y^2 + 2y + 1$	Complete the square.
$x - 3 = (\_\_\_\_\_)^2$	Simplify.
$\pm\sqrt{x-3} = y + 1$	Take the square root of each side.
$\_\_\_ \pm \sqrt{x-3} = y$	Subtract 1 from each side.
$f^{-1}(x) = -1 \pm \sqrt{x-3}$	Replace y with $f^{-1}(x)$.

Part B If necessary, restrict the domain of the inverse so that it is a function.

Because $f(x)$ fails the horizontal line test, $f^{-1}(x)$ is not a function. Find the restricted domain of $f(x)$ so that $f^{-1}(x)$ will be a function. Look for a portion of the graph that is one-to-one. If the domain of $f(x)$ is restricted to $[-1, \infty)$ then the inverse is $f^{-1}(x) = -1 + \sqrt{x-3}$.



If the domain of $f(x)$ is restricted to $(-\infty, -1]$, then the inverse is $f^{-1}(x) = -1 - \sqrt{x-3}$.

Example 4 Interpret Inverse Functions

TEMPERATURE **A formula for converting a temperature in degrees Fahrenheit to degrees Celsius is $T(x) = \frac{5}{9}(x - 32)$.**

Find the inverse of $T(x)$, and describe its meaning.

$T(x) = \frac{5}{9}(x - 32)$	Original function
$\_\_\_ = \frac{5}{9}(x - 32)$	Replace $T(x)$ with y.
$\_\_\_ = \frac{5}{9}(\_\_\_ - 32)$	Exchange x and y.
$\frac{9x}{5} = \_\_\_\_\_$	Multiply each side by $\frac{9}{5}$.
$\frac{9x}{5}\_\_\_\_\_ = y$	Add 32 to each side.
$T^{-1}(x) = \_\_\_\_\_$	Replace y with $T^{-1}(x)$.

$T^{-1}(x) =$ can be used to convert a temperature in degrees Celsius to degrees Fahrenheit.

Go Online You can complete an Extra Example online.

Watch Out!

Inverse Functions f^{-1} is read *f inverse* or *the inverse of f*. Note that -1 is not an exponent.

Go Online to see Part B of the example on using the graph of $T(x)$ and $T^{-1}(x)$.

Think About It!

Find the domain of $T(x)$ and its inverse. Explain your reasoning.

Think About It!

If $j(x)$ and $k(x)$ are inverses, find $[k \circ j](x)$.

Watch Out!

Compositions of Functions Be sure to check both $[f \circ g](x)$ and $[g \circ f](x)$ to verify that functions are inverses. By definition, both compositions must result in the identity function.

Go Online to see another example on verifying inverse functions.

Talk About It!

Find the domain of the inverse, and describe its meaning in the context of the situation.

Learn Verifying Inverses

Key Concept • Verifying Inverse Functions

Words: Two functions f and g are inverse functions if and only if both of their compositions are the identity function.

Symbols: $f(x)$ and $g(x)$ are inverses if and only if $[f \circ g](x) = x$ and $[g \circ f](x) = x$.

Example 5 Use Compositions to Verify Inverses

Determine whether $h(x) = \sqrt{x + 13}$ and $k(x) = (x - 13)^2$ are inverse functions.

Find $[h \circ k](x)$.

$[h \circ k](x) = h[k(x)]$	Composition of functions
$= h[\_\_\_\_\_]$	Substitute.
$= \sqrt{(x - 13)^2 \_\_\_\_\_}$	Substitute again.
$= \sqrt{x^2 - 26x \_\_\_\_\_ + 13}$	Distribute.
$= \sqrt{x^2 \_\_\_\_\_ + 182}$	Simplify.

Because $[h \circ k](x)$ is not the identity function, $h(x)$ and $k(x)$ are ________.

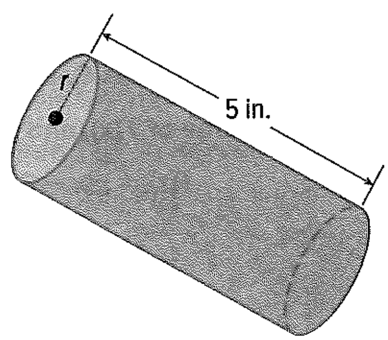
Check

Determine whether $f(x) = \frac{x}{9} + \frac{4}{3}$ and $g(x) = 9x + 12$ are inverses.

Explain your reasoning. ____________________

Example 6 Verify Inverse Functions

GEOMETRY **The formula for the volume of a cylinder with a height of 5 inches is $V = 5\pi r^2$. Determine whether $r = \sqrt{\frac{V}{5\pi}}$ is the inverse of the original function.**



Find $V \circ r$.	Find $r \circ V$.
$V = 5\pi r^2$	$r = \sqrt{\frac{V}{5\pi}}$
$= 5\pi\left(\sqrt{\frac{V}{5\pi}}\right)^2$	$= \sqrt{\frac{5\pi r^2}{5\pi}}$
$= 5\pi\left(\frac{V}{5\pi}\right)$	$= \sqrt{r^2}$
$= \_\_\_$	$= \_\_\_$

$r = \sqrt{\frac{V}{5\pi}}$ is the inverse of $V = 5\pi r^2$.

Go Online You can complete an Extra Example online.